THE
MATHEMATICAL GAZETTE.

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LONDON:
G. BELL AND SONS, LTD., PORTUGAL STREET, KINGSWAY,
AND BOMBAY.

VOL. VI. JULY, 1912. No. 99.

THE CAMBRIDGE SCHOOL OF MATHEMATICS.

CAMBRIDGE is honoured this year by its selection for the first meeting held in England by the International Congress of Mathematicians. The Cambridge School has played a notable part in the development of mathematics in Britain, and it may be of interest if I briefly summarize the leading facts of its history, which is indeed closely connected with that of the University. The School is of more than respectable antiquity and inevitably it has been sometimes flourishing and sometimes the reverse. Here I desire rather to bring out the salient features in its history than to discuss details or individual achievements.

The Medieval Period.—The University is among the oldest in Europe, having been founded about the close of the twelfth century. Its medieval curriculum and studies were much the same as those of other Universities of the time, and there is no need to describe them in this paper.

The Renaissance Period.—The modern development of the University begins with the Renaissance, which was warmly welcomed in Cambridge and was accompanied by a distinct development of mathematical teaching. I date the origin of its Mathematical School from this movement, and the first chapter of the history of the School may be said to cover the sixteenth

and the early years of the seventeenth century. It is true that, during the greater part of this period, no notable advance in the theory of the subject was made at Cambridge, or indeed in Britain—the first important British discovery in mathematics being that of logarithms, published by Napier in 1614—but it is worthy of remark that all the leading English mathematicians of the sixteenth century were educated at the University, and this fact even though their principal work was done elsewhere may justify my treating Cambridge as being then an important centre of mathematical teaching.

Among the earliest of these sixteenth century Cambridge students I chronicle, in passing, the names of Cuthbert Tonstall, 1474-1559, Robert Recorde, 1510-1558, John Dee, 1527-1608, and Thomas Digges, *circ.* 1530-1595, as representative writers of the time. A trifle later we come across Edward Wright, *circ.* 1558-1615, and Henry Briggs, 1561-1630, both of whom graduated in 1581 and subsequently accepted posts in London. Of these, Wright was a well-known authority on the theory of maps and navigation, and Briggs introduced the decimal notation as an operative form. The general adoption of logarithms was largely due to the efforts of the latter, and it was only the early death of Wright that prevented his sharing in the work. Towards the end of the period we find three men of greater mark in John Pell, 1611-1685, John Wallis, 1616-1703, and Seth Ward, 1617-1689. Much of their work was produced after the middle of the seventeenth century, and away from Cambridge, but it was there that they were made acquainted with and became interested in the subject. Was mathematics then widely studied in the University? It is difficult to say. Wallis, writing in 1635, when still in his teens, says mathematics "were then scarce looked upon as academical studies ... and among more than two hundred students ... in our college, I do not know of any two ... who had more of mathematics than I ... which was then but little ... for the study ... was at that time more cultivated in London than in the Universities." On the other hand, about 1639 we read that Ward had "brought mathematical learning into vogue in the University ... where he lectured his pupils in Master Oughtred's *Clavis*"; Pell, too, was teaching at Cambridge about this time, and his repute was such that he was asked by the authorities at Amsterdam to accept a mathematical chair there.

It will be noticed that the writers hitherto mentioned are now chiefly known by their work after they had left Cambridge. The explanation is that until this time the instruction in the subject in the Colleges was in general slight, while that in the University was carried on by young regent graduates under medieval conditions. In 1597 chairs of Geometry and Astronomy, under the

will of Sir Thomas Gresham, were established in London, and in 1619 Sir Henry Savile founded chairs on the same subjects at Oxford; other mathematical professorships had been founded even earlier on the Continent. These permanent and honourable posts attracted the best men, and the most brilliant students at the beginning of the seventeenth century—such as Briggs, Pell, Wallis, and Ward—were thus induced to leave for other Universities. It seemed as if there were no career for a mathematician at Cambridge.

The prosperity of a School with no established body of teachers is necessarily precarious, and, at the best, interest in the subject must be intermittent. Thus, when about 1641 the ablest mathematicians had left Cambridge, the study there flagged. At Oxford, where Ward and Wallis held chairs, and at London, it was otherwise: indeed it was from meetings in those cities that the Royal Society arose. The re-establishment of the supremacy of the Cambridge School was due to the influence of Newton, and the wide recognition of the value of his work.

The Newtonian Period.—At the Restoration there was a general rearrangement of things academical as well as political. Just at that time, in 1663, a professorship in mathematics was founded at Cambridge, and this promoted a revival of interest in the subject. Isaac Barrow was the first occupant of the chair. His lectures—on the principles of the subject, geometrical optics, and properties of curves—are extant, but to his disappointment the attendance at them was small. He was however fortunate in having among his pupils Isaac Newton, in whose favour, in 1669, he resigned the chair, thus securing to Newton, when still under twenty-seven, the opportunity to prosecute and promulgate his discoveries.

It so happens that we know the exact course of reading pursued by Newton in his student days, and the line of his early investigations. The former shows what text-books were then available to students, and by way of parenthesis I describe it. Newton came into residence in 1661, when just under the age of nineteen, ignorant of mathematics, and with no intention of taking up the subject. At a Fair at the beginning of term he chanced to buy a book on Astrology, but his ignorance of Geometry and Trigonometry prevented his understanding it. Thereupon he bought Euclid's *Elements* and Oughtred's *Clavis*, both of which he mastered with ease. He then bought Descartes's *Geometry*, which he read, though with considerable difficulty. So far he had had no assistance, but he now definitely took up mathematical studies, and (presumably) worked under Barrow's directions, reading Kepler's *Optics*, Vieta's *Works*, Schooten's *Miscellanies*, Wallis's *Arithmetica Infinitorum*, and again Descartes's *Geometry*.

He graduated in January, 1665 (N.S.), and was then free to follow his own inclinations. Within the next two years he laid the foundations of much of his future work. He enunciated or used the binomial theorem in April 1665, direct fluxions in November 1665, his theory of colours in January 1666, inverse fluxions in May 1666, and later that year convinced himself that gravity was the force that kept the planets in their orbits, for "in those days," he wrote later, "I was in the prime of my age for inventions." Probably he took pupils, and his note-books were open to them. In Cambridge his reputation was already established.

Newton was elected professor in 1669. His lectures for eighteen out of the first nineteen years of the tenure of the chair are extant. They are chiefly concerned with algebra (universal arithmetic), optics, and astronomy. They were given once a week for one term in the year and each lasted about half an hour, being dictated* as rapidly as his audience could take it down. More important than his formal lectures was the fact that he was accessible to students during two hours on two days every week in term, and on one day every week in vacation when in residence: it was then that he explained difficulties and amplified his exposition. He had exceptional command over the processes of geometry, algebra, and fluxions, but I venture to say that he valued mathematics chiefly as a means of attacking physical problems, and that it was to his theories of geometrical optics, light, and gravitation that he specially desired to attract the attention of his followers.

I need not treat of the publication of his works and the gradual acceptance of his theories. In Cambridge his views were, from the beginning, generally adopted. It was no doubt a tribute to his reputation that a second mathematical chair—devoted to Astronomy and Experimental Philosophy—was founded in 1704; and that in 1710, Lady Sadleir endowed various College lectureships on Algebra and its applications.

I note in particular among Newton's pupils the names of Roger Cotes, 1682-1716, and Robert Smith, 1689-1768, who occupied in succession the Plumian Chair, and Brook Taylor, 1685-1731; and as representatives of the School in England, the names of David Gregory, 1661-1708, Humphry Ditton, 1675-1715, Abraham Demoivre, 1667-1754, and Colin Maclaurin, 1698-1746. The School started with everything in its favour, but its early promise was not fulfilled. It is well known that most bitter feelings were aroused in the controversy whether Leibnitz, before developing the Calculus, had obtained the fundamental ideas of it from Newton or had originated it independently. One result of

*In the Library of Trinity College there is a manuscript volume by Cotes, which probably represents notes thus taken down.

the quarrel was the separation, under a keen sense of injustice, of British mathematicians from their continental contemporaries. Intercourse and exchange of views are essential to vigour, and the more varied the outlook and training of those concerned, the more fruitful is the intercourse. The self-inflicted isolation of the Newtonian School sufficiently accounts for its rapid decadence. The effect was intensified by the manner in which its members confined themselves to geometrical demonstrations. If Newton gave geometrical proofs in the *Principia*, it was because their validity was unimpeachable, and since his results were novel, he did not wish the discussions as to their truth to turn on the methods used to demonstrate them. But his followers long after the principles of the Calculus had been accepted, continued to employ geometrical proofs whenever it was possible. No considerable body of important discoveries were made by them after his death, which we may accordingly take as marking the end of this period in the history of Cambridge mathematics.

The Eighteenth Century.—During the century following the death of Newton, the work produced at Cambridge was unimportant. There were already two professorships in mathematics: additional chairs were founded, one in 1749 by Thomas Lowndes in Astronomy and Geometry, and another in 1783 by Richard Jackson in Natural and Experimental Philosophy and Chemistry. Most of the professors were however undistinguished, and indeed but little interested in the subject. Then, and well into the following century, a mathematical chair was often regarded only as a prize or a means of securing leisure, and at best, merely as offering a position where a man could pursue his own researches undisturbed by other duties. Notwithstanding this, Cambridge remained the centre of mathematical studies in Britain. Teaching in the subject was excellently organized, and the number of students in it steadily increased. This was due partly to the immense influence exerted by the Colleges and by "Pupil-mongers," but in the latter half of the century is mainly attributable to the development of a rigorous system of examination in mathematics which for a long time formed the chief avenue to University distinctions.

To a foreigner the maintenance of a School by such means must seem paradoxical, but Collegiate Universities, like Oxford and Cambridge, are complex institutions. Every student of the University was a member of a College; it was from his College or from private teachers that he obtained instruction, and it was to his College that he looked for reward and encouragement. The University did not teach him, and beyond seeing that he kept certain statutable exercises for his degree, did little for him. The most important of these exercises were "acts" (theses, and disputations on them), which had to be kept shortly before the

degree was conferred; the men being admitted to their degrees in an order corresponding roughly to their performances. At the close of the seventeenth century these theses were usually on philosophical questions, the subjects being proposed by the candidates and approved by the proctors or moderators.

It was necessary to prepare carefully for these acts, which made considerable demands on the ability and readiness of candidates. This preparation was the business of College tutors and pupil-mongers, who directed the studies of their pupils for the purpose. The exact status of these pupil-mongers is not known, nor is it certain how far and for what consideration they took pupils outside their own Colleges, but even though we do not know the exact conditions under which they worked, we shall not be far wrong in describing them as private tutors or "coaches." College tutors and pupil-mongers, whether mathematical or not, were deeply impressed by the system of philosophy expounded by Newton, and probably by the end of the seventeenth century it was not unusual to require students to make themselves acquainted in general with his conclusions. Bentley's lectures in 1696 indicate this tendency, and two years earlier, in 1694, Richard Laughton of Clare, the best known pupil-monger of the time, induced a student to submit a thesis on a subject connected with the *Principia*. The younger tutors soon went further, and under their influence mathematics became a usual subject for study: most likely, however, mainly with a view to its applications, and not as an instrument of education. In 1710 the movement had developed so far that Laughton, presiding in the Schools as proctor, persuaded a student to keep one act (out of three) on a strictly mathematical question, and this soon became the customary course.

The normal academic course before the first degree lasted three years and a half. We possess courses of study drawn up by two well-known tutors of the period, showing how this time might or should be occupied. They illustrate what an important part in it mathematics already played. Thus Daniel Waterland of Magdalene, in 1706, gives the following as a normal course of reading in "philosophical studies": in the first year, Arithmetic, Euclidean Geometry, Algebra, Logic, Geography, and Trigonometry; in the second year, Conic Sections, Mechanics, Easy Astronomy, Physics (Rohault), and Philosophy (Locke and Cheyne); in the third year, Elementary Optics, Law and Political Science (Grotius and Puffendorf), Ethics, and Theology; and in the last half-year, Optics, Astronomy, and Metaphysics. Students who continued in residence and studied mathematics further were then advised to read Newton's *Principia*, Ozanam's *Cursus*, Sturmius's *Works*, Huyghens's *Works*, Newton's *Algebra*, and Milne's *Conic Sections*. Text-books were recommended; in later editions they

were altered, but no material changes were introduced in the course, which was current for half a century. A somewhat similar scheme of study was sketched out in 1707 by Robert Green of Clare, who was deemed an old-fashioned teacher. He recommended his pupils to spend their first year in the study of Classics; the second on Logic, Ethics, Arithmetic, Algebra, Geometry and Corpuscular Philosophy; the third on Analytical Geometry and Natural Science; the fourth on Logarithms, Trigonometry, Fluxions and Infinite Series, Mechanics of Solids and Fluids, and Astronomy. Statements published, widely circulated, reprinted, and not criticized at the time, must be received with respect, but I think these courses represent ideals rather than actual performances, and I cannot believe that either of them was often taken in its entirety by ordinary students. For the range of knowledge implied we cannot rely entirely on the text-books then current, for mathematical tutors, at that time, as later, often used manuscript commentaries embodying traditional methods of proof thus handed on from one generation to another. At a later period the use of such manuscripts became a normal part of the teaching, and this lasted to a time well within my own memory.

From the earliest days the proctors had the power, though the opportunity was not always used, of questioning a candidate at the end of a disputation, but the language used was Latin. Shortly after 1710 there were some difficulties in conducting the exercises and, to give an opportunity of comparing the men, the proctors and moderators summoned all the candidates to the Senate-House, and there publicly examined them, questions and answers being given in English. The advantages of the scheme were so patent that it was retained, and by 1730 may be considered to have been definitely established. With the introduction of an examination of all candidates together, closer attention was paid to securing a strict order of merit, and more confidence was felt in the published order. The Senate-House Examination, later known as the Mathematical Tripos, was partly on philosophy and partly on mathematics, but the latter soon became the essential part. By 1750 the examination had extended to two days and a half. In 1763 the traditional rules for conducting it took definite shape, and the previous disputations were used chiefly to get a rough preliminary order. About 1770 the answers were given in writing, and it was now frankly recognized that the examination was strictly competitive. About 1790 some of the more important papers of questions were printed. In 1799 a pass standard was introduced, and by this time the final Tripos order was settled only by the examination. An important part of the system consisted in papers of problems on all the subjects then read, and designed to encourage originality.

In the latter half of the century the examination had become a contest carried on under strict rules in which success was keenly sought. The publication of the final class list was one of the chief events of the year. Success in it was regarded as the crown of the academic course, and brought with it, in the shape of a fellowship, an immediate competence and a reasonable prospect of an assured career. As a high place in the final list meant so much to a successful candidate, a class of private tutors arose who made it their special business to prepare their pupils for it. They are the successors of the pupil-mongers of an earlier period. The earliest indisputable instances I can quote are those of John Wilson, whose name is still known by its association with a proposition in the Theory of Numbers, and Robert Thorp. The latter is described, about 1761, as being "of eminent use to young men in preparing them for the Senate-House Examinations and peculiarly successful." And it was added that "one young man of no shining reputation with the assistance of Mr. Thorp's tuition had stood at the head of wranglers." In a grace of the Senate, passed in 1781, it is stated that almost all sophs then resorted to private tuition, and for more than a century subsequently, the practice was well established. These were the men who really directed the reading of the students. Even non-residents, if reputed to be successful coaches, drew pupils. Thus John Dawson, a medical practitioner in the Lake District, regularly prepared pupils in the vacations for the Senate-House Examination, and at least eleven of the senior wranglers between 1781 and 1800 are known to have studied under him.

As I have already remarked, the contributions of the Cambridge School to the progress of the subject during the last three quarters of the eighteenth century were unimportant.

The Nineteenth Century.—The prominent features of the history of the Cambridge School of Mathematics during the nineteenth century are the further development of the system of coaching and the confinement of the subjects studied to those scheduled in the Tripos regulations, accompanied by a striking revival of interest in the subject, and the appearance of a remarkable group of mathematical physicists.

During the early years of the century the evil consequences to the School of its isolation and its use only of Newtonian methods were recognized. Under the influence of Robert Woodhouse, George Peacock, Charles Babbage, and John Herschel, text-books, characterised by a free use of analysis, were produced, and in 1817 the symbols of differentiation were introduced in the papers set in the Senate-House Examination; thenceforward analytical methods were especially cultivated. But the desire to apply mathematics to the problems of the external world soon became apparent, and the physical researches to which I allude later

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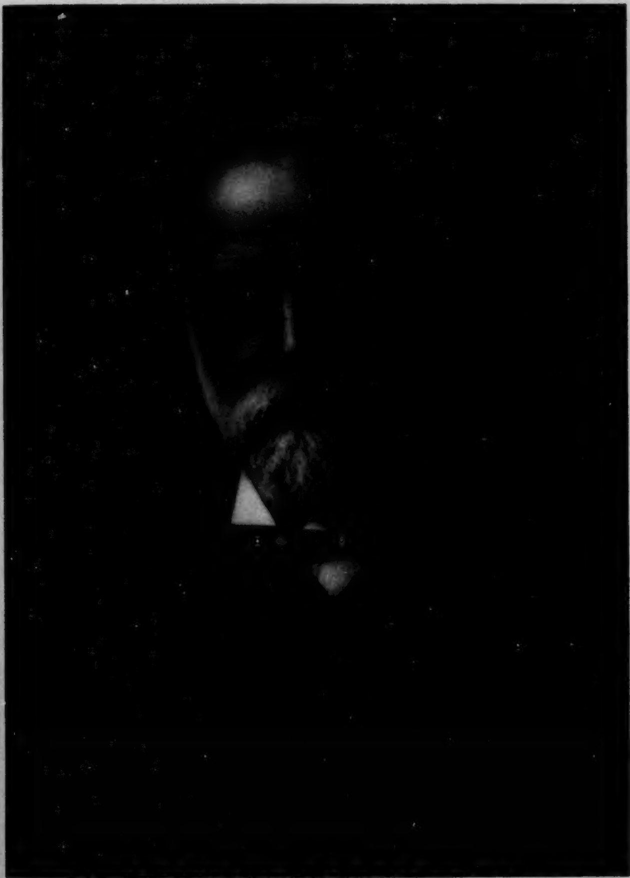

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SIR GEORGE DARWIN, K.C.B., F.R.S.

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will always be regarded with pride as a remarkable feature in the history of the University. Until the last quarter of the century, mathematics remained the dominant study of the place, and for many years the number of undergraduates, reading the subject at any one time, rarely fell short of three hundred.

During this century Cambridge continued to be the principal, though not the sole, centre of mathematical studies in Britain. There were already three professorships in the theory of the subject. In addition to these a chair of Pure Mathematics was founded in 1863; one of Experimental Physics, with a Laboratory attached, in 1871; and one of Mechanism and Applied Mechanics (for Engineering), with a Laboratory and Shops attached, in 1875. Further in 1882 five University Lectureships were established at the expense of the Colleges. But, save in Experimental Physics and Engineering, professors, distinguished though many of them were, were as little in touch with undergraduates taking the normal course as their predecessors in the eighteenth century. In 1849 a Board was appointed to supervise mathematical studies, and it was hoped that one effect would be to bring the professors into closer touch with students, but in this it failed, and throughout this period Cambridge remained almost unique in having a number of eminent mathematical professors with whom the great majority of the mathematical students before graduation were wholly unacquainted; this was not the case in other faculties in Cambridge. During the last twenty years of this century students who continued to read the subject after graduation began, under College pressure, to come in touch with the professors and University lecturers. Still, had there been no chair of mathematics in the University it is probable that the history of the School would have been practically unaltered, all undergraduate teaching having been in effect given by College lecturers and private tutors. The advance of knowledge is a primary duty of professors, but their influence is rarely effective unless they are also active and fairly efficient teachers.

Early in the nineteenth century the Senate-House Examination was confined to mathematics, and became technically known as the Mathematical Tripos. Throughout the century constant efforts were made, with reasonable success, to keep the subjects of examination abreast of the times, and especially to encourage the reading of mathematical physics. Until 1882 the course before the first degree lasted about three years and a half. After 1882 it lasted three years, but a second examination was held in the following year for students who had specialized in higher subjects selected by themselves. To cover in this time the subjects before graduation demanded strenuous and continuous work. It is easy to pick holes in any elaborate scheme, but it may fairly be said that the system ensured a thorough and

accurate grounding in all the then usual branches of mathematics and their ordinary applications, and that by the time of their first degree men who had gone through it were well equipped to take up higher studies or research, if their inclinations led them that way, while in any case they had had a severe logical training which was an excellent preparation for professional careers. On the other hand, it was objected that the incessant polishing of the instruments to be used later was as weary as it was unnecessary, that the mental discipline was too severe and too narrow, and that an examination test gave unreasonable encouragement to the cultivation of rapidity, while the straining to combine conciseness with accuracy disturbed the due presentation of the subject. Be this as it may, the Tripos and its regulations dominated the situation, and Cambridge mathematics and mathematicians of the nineteenth century were the direct product of the system. Judged by the output, I do not think it can be said to have resulted in failure; and may-be Cayley, Sylvester, Adams, Green, Stokes, Kelvin, and Maxwell—to mention no others—were none the worse for having been compelled to study seriously subjects other than those in which they were most interested.

From its foundation a distinctive feature of the Tripos had been the production of a list arranged in order of merit of those who passed, and for most of the time this feature had been highly prized. Towards the end of this period the maintenance of this order became a subject of controversy. That it ensured a thorough grounding and wide reading, compelled industry, and that the men were placed and could be placed in proper order according to the conditions of the examination was not denied. It was however urged that competition necessarily led candidates to attach undue importance to the order in the final list, thus inducing them to work too hard and to confine their reading within the limits of the schedules of the Tripos, and that its retention brought out exactly those results to which objection was taken, as stated above. Finally, but not until 1909, order of merit was replaced by an alphabetical arrangement in classes.

A remarkable feature of this period was the fact that until the closing years of the century, nearly every student read with a private tutor, and that two or three private tutors obtained in succession almost a monopoly of the direction of the studies of advanced honours students. Thus William Hopkins, 1793-1866, in the twenty-two years from 1828-1849, had among his pupils 175 wranglers, of whom 17 were senior, 44 in one of the first three places, and 108 in one of the first ten places. So again Edward John Routh, 1831-1907, in the thirty-one years from 1858-1888, had between 600 and 700 pupils, most of whom became wranglers, 27 being senior in the Tripos and 41 Smith's

prizemen. To organize teaching on this scale demanded rare gifts.

Perhaps it may be of interest to describe, by way of example, the general features of Routh's system. He gave catechetical lectures three times a week to classes of eight or ten men of approximately equal knowledge and ability. The work to be done between two lectures was heavy, and included the solution of some eight or nine fairly hard examples on the subject of the lectures. Examination papers were also constantly set on Tripos lines (book-work and riders), while there was a weekly paper of problems set to all pupils alike. All papers sent up were marked in public, the comments on them in class were generally brief, and, to save time, solutions of the questions were circulated in manuscript. Teaching also was supplemented by manuscripts on the subjects. Finally to the more able students he was accustomed, shortly before their Tripos, to give memoirs or books for analyses and commentaries. The course for the first three years and the two earlier long vacations covered all the subjects of the Tripos—the last long vacation and the first term of the fourth year were devoted to a thorough revision. Of what is called cramming there was no trace; Hopkins and Routh might say that a particular demonstration was so long that it could not be required in the Tripos, but none the less they expected their pupils to master it. The system had faults, but at any rate it was under Hopkins and Routh that nearly all the best-known representatives of Cambridge mathematics in the nineteenth century were educated. The effectiveness of teaching of this kind was dependent on intimate constant personal intercourse, and the importance of this cannot be overrated. The scandal of the system consisted in the fact that the men were compelled to pay heavy fees to the University and Colleges for instruction, and yet found it advantageous to go elsewhere at their own expense to get it. During the last quarter of the nineteenth century College lecturers began to share with the Coaches the general direction of studies. Post-graduate work has also to some extent been brought under the influence of professors and University lecturers—these not uncommonly suggesting subjects for dissertations for fellowships, Smith's Prizes, etc. But the students thus influenced are not numerous, and it still remains true that the majority of mathematical undergraduates are so out of touch with the professors in the subject as to be unacquainted even with their personal appearance.

I have already alluded to the brilliant achievements of the Cambridge mathematical physicists of this period. The beginning of this movement may be traced to Airy and his contemporaries, it was developed by Green, and we may consider it definitely established by the labours of Stokes, with whom I associate the

names of Kelvin and Maxwell, their experiments being combined with mathematical investigations. Maxwell's work at the Cavendish laboratory has been carried on by Lord Rayleigh, Sir Joseph Thomson, and numerous investigators working there under them—the teaching and inspiration here being directly due to the professors. The resulting School seems to me to be the direct successor of the Newtonian School of the seventeenth century, and I do not doubt that were Newton here now it is with the Cavendish laboratory that he would wish to be particularly associated. But this branch of Cambridge mathematics is still making history, and therefore I content myself with only a brief reference to it. This work is, I think, the most striking achievement of the School during last century; I may also mention, as directly due to it, the production of a large number of excellent text-books, and in recent years some valuable work on the fundamental assumptions and principles of pure mathematics.

The Present Period.—In this article I have not unnaturally avoided mentioning the work of those who fortunately are with us to-day, and for similar reasons I do not propose to say anything about the progress of the School in the opening years of the twentieth century. The reconstitution in 1909 of the Tripos, and the destruction of many of the distinctive features of the former scheme must profoundly modify the future history of mathematics at Cambridge, and perhaps the long-continued efforts to bring students into closer touch with professors and lecturers may be at last crowned with success. The change in the Tripos regulations has been accompanied by a curious alteration in the popular subjects, and to-day but few of the young graduates who desired the change are interesting themselves in those branches of applied mathematics once so generally studied, but rather are turning their attention to subjects like the Theories of Functions and Groups. It is too early to say whether this is only a passing movement.

By way of supplement to the foregoing account, I append a list of those who have held the various University chairs and lectureships. In a collection of portraits of mathematicians which I possess, I have likenesses of nearly all the holders of these posts, and shall be pleased to show them to any Members of the Conference who may care to look at them.

W. W. Rouse Ball.

The *Lucasian Professorship of Mathematics* was founded in 1663 by Henry Lucas. The successive occupants of the chair have been: Isaac Barrow, 1664-1669; Isaac Newton, 1669-1702; William Whiston, 1702-1711; Nicholas Sanderson, 1711-1739; John Colson, 1739-1760; Edward Waring, 1760-1798; Isaac Milner, 1798-1820; Robert Woodhouse, 1820-1822; Thomas Turton, 1822-1826; George Biddell Airy, 1826-1828; Charles Babbage, 1828-1839; Joshua

King, 1839-1849; George Gabriel Stokes, 1849-1903; Joseph Larmor, 1903 *et seq.*

The *Plumian Professorship of Astronomy and Experimental Philosophy* was founded in 1704 by Thomas Plume. The successive occupants of the chair have been: Roger Cotes, 1707-1716; Robert Smith, 1716-1760; Anthony Shepherd, 1760-1796; Samuel Vince, 1796-1822; Robert Woodhouse, 1822-1828; George Biddell Airy, 1828-1836; James Challis, 1836-1883; George Howard Darwin, 1883 *et seq.*

The *Lowndean Professorship of Astronomy and Geometry* was founded in 1749 by Thomas Lowndes. The successive occupants of the chair have been: Roger Long, 1750-1771; John Smith, 1771-1795; William Lax, 1795-1836; George Peacock, 1836-1858; John Couch Adams, 1858-1892; Robert Stawell Ball, 1892 *et seq.*

The *Sadlerian Professorship of Pure Mathematics* was founded in 1863 from a benefaction given in 1710 by Lady Sadleir. The successive occupants of the chair have been: Arthur Cayley, 1863-1895; Andrew Russell Forsyth, 1895-1910; Ernest William Hobson, 1910 *et seq.*

The *Cavendish Professorship of Experimental Physics* was founded in 1871 by the University; the laboratory attached being built at the expense of the then Chancellor, the Duke of Devonshire. The successive occupants of the chair have been: James Clerk Maxwell, 1871-1879; John William, Baron Rayleigh, 1879-1884; Joseph John Thomson, 1884 *et seq.*

The *Professorship of Mechanism and Applied Mechanics*, with laboratories and shops attached, was founded by the University in 1875. The successive occupants of the chair have been: James Stuart, 1875-1890; James Alfred Ewing, 1890-1903; Bertram Hopkinson, 1903 *et seq.*

Five *Lectureships in Mathematics* were created in 1882 under the directions of Royal Commissioners, and subsequently two others tenable, if desired, with one of the above, were founded. The successive holders have been: Joseph John Thomson, 1884; Andrew Russell Forsyth, 1884-1895; William Herrick Macaulay, 1884-1887; Richard Tetley Glazebrook, 1884-1898; Ernest William Hobson, 1884-1910; Joseph Larmor, 1885-1903; Richard Pendlebury, 1888-1901; Henry Frederick Baker, 1895 *et seq.*; Augustus Edward Hough Love, 1898-1899; Hector Munro Macdonald, 1899-1904; Herbert William Richmond, 1901 *et seq.*; George Ballard Mathews, 1903-1905; James Hopwood Jeans, 1904-1906, 1910-1912; John Gaston Leathem, 1905 *et seq.*; Robert Alfred Herman, 1906 *et seq.*; Edmund Taylor Whittaker, 1905-1906; Thomas James I'Anson Bromwich, 1909 *et seq.*; John Hilton Grace, 1910 *et seq.*

The Mathematical Association.

LONDON BRANCH.

PRESIDENTIAL ADDRESS ON THE THEORY OF PROPORTION.

By Professor M. J. M. Hill, M.A., LL.D., Sc.D., F.R.S.

10th February, 1912.

INTRODUCTORY.

1. I desire in the first place to express my thanks to the members of the London Branch of the Mathematical Association for the honour they have done me in electing me to the office of president. I esteem it a privilege to take part in the efforts the Association is making to bring about improvements in the methods of teaching Mathematics.

In what position does the work of the Association now stand? Is it in fact in the position described by Sir J. J. Thomson in his address to the Association of Public School Science Masters? He is reported to have said that he had come to the conclusion that if you have intelligent masters and small classes it does not matter much what theory of education you adopt, and if you have not these, well, it does not much matter either.

That seems to me to be a counsel of despair. I prefer to take my stand by the side of Professor Hobson, who said two notable things in his recent address on "The Democratisation of Mathematical Education."

The first was that the business of this Association was "the progressive adaptation of methods and matter of teaching to meet the needs of those who lacked any exceptional capacity."

The second was that "Education in Mathematics must be pronounced a failure if it did not rise beyond the purely practical aspect to the domain of principle. The most important educational aspect of the subject was an instrument for training boys and girls to think accurately and independently."

The Subject Selected.

2. I have chosen as my subject one which will illustrate both of Professor Hobson's remarks, *i.e.* I have taken a definite portion of the curriculum, which has been regarded hitherto as the exclusive possession of the Mathematical Aristocracy, and I propose to throw it open to the Mathematical Democracy; and I have chosen a subject which is in a high degree fitted to promote accurate thinking. I hope that I may thereby make a contribution to the constructive work of the Association. The subject is one which has exerted a strange fascination over me for years, a fascination which has increased the more I worked at it.

It was formerly, at least in name, a part both of School and University Curricula, but it has now, in the general revolt against Euclidean methods, disappeared from the one without being assigned a definite place in the other. I refer to the Theory of Proportion, so far as it deals with Incommensurable Magnitudes.

To this subject Euclid devoted the Fifth Book of his *Elements*, the substance of which is usually attributed to Eudoxus of Cnidus. His treatment is so difficult that even when he reigned supreme over English Mathematical Education, it was usual simply to pass over it, with the exception of his Fifth Definition, and to give *so-called* algebraic proofs* of the properties of equal

* These so-called algebraic proofs are applicable only to ratios of commensurable magnitudes. The proofs in the Fifth Book (which is a treatise on Algebra and not on Geometry) are applicable to the ratios of Incommensurable Magnitudes.

ratios which are required for the propositions of the Sixth Book. The definition retained was learned by rote, and most probably not understood. I must not be understood to complain of the omission of the rest of the book, because the text, in the form in which we have it, is far beyond the intelligence not merely of the average school boy, but of almost all clever ones as well.

At the same time an exposition on these lines is not logically defensible, and as soon as it became my duty to be responsible for teaching the subject, after rejecting other methods which I had tried and found wanting, I directed my efforts to the simplification of Euclid's argument in the hope that this could be done without impairing its rigour.

3. I am well aware that in discussing this immortal work I stand on difficult ground.

Saccheri called attention to a departure from logical rigour in the proof of the 18th proposition, and Simson to one in that of the 10th, as these have come down to us.

After giving proofs of these propositions not open to criticism, Simson says:

"The Fifth Book being thus corrected, I most readily agree to what the learned Dr. Barrow says, 'that there is nothing in the whole body of the elements of a more subtile invention, nothing more solidly established, and more accurately handled than the doctrine of proportionals.'"

Sir Thomas Heath, in his great edition of Euclid, Vol. II. p. 186, published in 1908, quoting Simson's words, adds:

"Simson's claim herein will readily be admitted by all readers who are competent to form a judgment upon his criticisms and elucidations of Book V."

And that great master of Logic, Augustus de Morgan, has said in his *Treatise on the Connexion of Number and Magnitude*, p. 1:

"This same book and the logic of Aristotle are the two most unobjectionable and unassailable treatises which ever were written."

Perhaps it may be said of De Morgan as of Dean Middleton, that, though he was so wary held and wise, that as 'twas said he scarce received for Gospel what the Church believed, he had a superstition of his own.

What that superstition was I will disclose later.

And indeed, apart from the two points noted by Saccheri and Simson, I believe the logic of Euclid's argument is unassailable.

Nor will I dispute its subtlety, for which I have the most unbounded admiration.

Yet I think, as has been somewhere said, that the greatest honour that can be paid to a great man is the patient and reverent study of his work; and I hold also that this does not mean that it is to be regarded as a perfect whole which ought not to be either touched or handled, for Science is neither finished nor finite. With Rabbi ben Ezra we may say that it is

> "better youth
> Should strive, through acts uncouth
> Toward making, than repose on aught found made."

4. Therefore I shall venture to state an objection to the arrangement of the argument, which was published for the first time in the *Cambridge Philosophical Transactions*, Vol. XVI., in 1897.

This criticism was acknowledged as valid by the German reviewer in the *Fortschritte der Mathematik*,* but has passed without notice in England and America. In the arrangement of the argument I propose, the reasoning is simplified without loss of rigour. When so re-arranged I have found no difficulty during the past twelve years in explaining it to students who have a working acquaintance with Elementary Algebra.

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* Bd. 28, p. 152, 1897.

It was not until I discovered the reason for the difficulty in Euclid's presentation of the argument that I found that this could be done. In order to explain this, let me first give some account of the work.

THE FIFTH BOOK OF EUCLID'S ELEMENTS.

I. Definitions.

5. The most important definitions are the 3rd, the 4th, the 5th, and the 7th.

(*a*) The *third* definition is translated thus by De Morgan:

Ratio is a certain mutual habitude* of two magnitudes of the same kind depending upon their quantuplicity.†

(*b*) The *fourth* definition is translated thus by De Morgan:

"Magnitudes are said to have a ratio to each other which can, being multiplied, exceed 'one the other.'"

(*c*) The *fifth* definition (the test for determining if two ratios are equal):

If A, B, C, D are four magnitudes, then A has the same ratio to B as C has to D, if when any integers *whatever*, r, s have been chosen, the following sets of conditions are satisfied:

(i) If the integers r, s are such that $rA<sB$, it is necessary that $rC<sD$.

(ii) If the integers r, s are such that $rA=sB$, it is necessary that $rC=sD$.

(iii) If the integers r, s are such that $rA>sB$, it is necessary that $rC>sD$.

(It can be proved that the second set of conditions must be satisfied if the first and third sets hold good.)

(*d*) The *seventh* definition (the test for distinguishing the greater of two *Unequal* Ratios from the smaller):

The ratio of A to B is greater than that of C to D if a single pair of integers, r, s, can be found such that

if $rA>sB$, then either $rC=sD$ or $rC<sD$.

(It can be proved that if integers r, s exist such that $rA>sB$, $rC=sD$, then other integers r', s' exist such that $r'A>s'B$, $r'C<s'D$. Consequently it is possible to leave out of consideration the alternative $rC=sD$.)

II. Propositions (First Group). Nos. 1, 2, 3, 5, 6.

Denoting positive integers by small letters and magnitudes by large letters these propositions are:

1. $r(A+B+C+\ldots)=rA+rB+rC+\ldots$.
2. $(a+b+c+\ldots)R=aR+bR+cR+\ldots$.
3. $r(sA)$ is the same multiple of A as $r(sB)$ is of B.
5. If $A>B$, then $r(A-B)=rA-rB$.
6. If $a>b$, then $(a-b)R=aR-bR$.

These are simple cases of the application of the Associative and Distributive Laws, and involve only multiples of magnitudes, not ratios of magnitudes.

With this group it is convenient to include the Proposition

$$r(sA)=s(rA),$$

which may be obtained from Prop. 1. It is a case of the Commutative Law.

* σχέσις, method of holding or having, mode or kind of existence.

† πηλικότης, for which there is no English word; it means relative greatness, and is the substantive which refers to the number of times or parts of times one is in the other.

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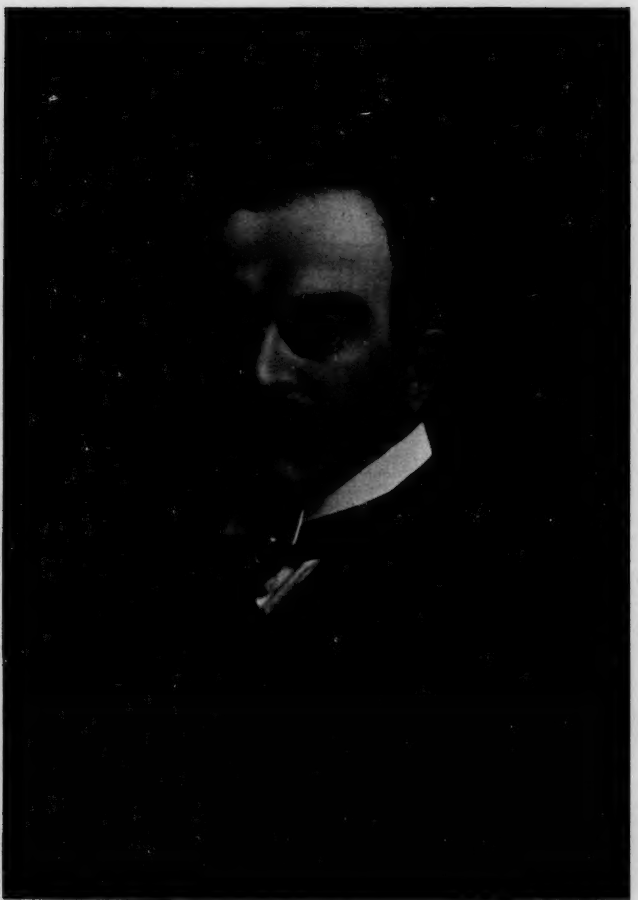

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Facing p. 326.

III. PROPOSITIONS (Second Group). Nos. 4, 7, 11, 12, 15 and 17.

These propositions express properties of Equal Ratios, and are deduced by Euclid *directly* from the Fifth Definition.

(The sign of equality will be used in place of the sign :: for 'is the same as.')

4. If $A:B=C:D$, then $rA:sB=rC:sD$.
7. If $A=B$, then $A:C=B:C$ and $C:A=C:B$.
11. If $A:B=C:D$, and if $E:F=C:D$, then $A:B=E:F$.
12. If $A:B=C:D=E:F$, and if all the magnitudes are of the same kind, then $$A:B=A+C+E:B+D+F.$$
15. $A:B=nA:nB$.
17. If $A+B:B=C+D:D$, then $A:B=C:D$.

With this group may be associated the following propositions from the Tenth Book:

Euc. X. 5. $a:b=aN:bN$.

Euc. X. 6. If $a:b=X:Y$, then $X=aG$, $Y=bG$.

Euc. X. 7. Incommensurables are not to one another as whole numbers.

IV. PROPOSITIONS (Third Group). Nos. 8, 10, 13.

8. (i) If $A>B$, then $A:C>B:C$.
   (ii) If $A<B$, then $C:A>C:B$.
10. (i) If $A:C>B:C$, then $A>B$.
   (ii) If $C:A>C:B$, then $A<B$.
13. If $A:B=C:D$, and if $C:D>E:F$, then $A:B>E:F$.

These express properties of Unequal Ratios, and their proofs depend on the Seventh Definition.

Euclid employs them to prove properties of Equal Ratios and for this purpose only.

V. PROPOSITIONS (Fourth Group). Nos. 9, 14, 16 and 18–25.

9. (i) If $A:C=B:C$, then $A=B$. (ii) If $C:A=C:B$, then $A=B$.
14. If A, B, C, D are magnitudes of the same kind, and if $A:B=C:D$, then $B \gtreqless D$ according as $A \gtreqless C$.
16. If A, B, C, D are magnitudes of the same kind, and if $A:B=C:D$, then $A:C=B:D$.
18. If $A:B=C:D$, then $A+B:B=C+D:D$.
19. If $A+C:B+D=C:D$, then $A:B=A+C:B+D$.
20. If $A:B=T:U$, and if $B:C=U:V$, then $T \gtreqless V$ according as $A \gtreqless C$.
21. If $A:B=U:V$, and if $B:C=T:U$, then $T \gtreqless V$ according as $A \gtreqless C$.
22. If $A:B=T:U$, and if $B:C=U:V$, then $A:C=T:V$.
23. If $A:B=U:V$, and if $B:C=T:U$, then $A:C=T:V$.
24. If $A:C=X:Z$, and if $B:C=Y:Z$, then $A+B:C=X+Y:Z$.
25. If A, B, C, D are four magnitudes of the same kind, and if $A:B=C:D$, and if A be the greatest of them, then $A+D>B+C$.

All these propositions deal with properties of *equal* ratios, but Euclid's proofs depend directly or indirectly on Props. 8, 10, 13, and therefore ultimately on the 7th definition, so that the proofs depend on properties of *unequal* ratios.

Their proofs can, however, be obtained from the Fifth Definition, without using the Seventh Definition, and then 14 can be deduced immediately from 16, 20 from 22 and 21 from 23.

THE DEFINITIONS.

6. Most of the long list of definitions with which the book opens are verbal, and I have therefore selected for special mention only the 3rd, 4th, 5th and 7th.

THE THIRD DEFINITION.

7. I have given above De Morgan's translation. He says (*l.c.* p. 63, line 4): "Ratio is relative magnitude."

If we say that the ratio of A to B is the measure of the relative magnitude of A as compared with B, I think that this will give us all that can be extracted from Euclid's third definition. *The important thing is to show how the measure is determined*, and this I do later.

THE FOURTH DEFINITION.

8. I have given De Morgan's translation above. It is sometimes rendered, "Two magnitudes are said to have a ratio when the less can be multiplied so as to exceed the greater."

But De Morgan contends that this does not represent the meaning of the Greek original.

Whichever form be accepted, the Axiom of Archimedes is assumed, viz.: If A and B are two magnitudes of the same kind, it is always possible to find a multiple of the less which will exceed the greater.

The third and fourth definitions may be regarded as doing two things:

In the first place they call attention to important properties of two magnitudes *of the same kind*. The idea of magnitudes *of the same kind* has to be introduced into the argument, and in order to do it, it is necessary to define the characteristics possessed by such magnitudes.

MAGNITUDES OF THE SAME KIND.

9. A system of magnitudes is said to be *of the same kind* when they possess the following characteristics:

(1) Any two magnitudes *of the same kind* may be regarded as equal or unequal.

In the latter case one of them is said to be the smaller, and the other the larger of the two.

(2) Two magnitudes *of the same kind* can be added together. The resulting magnitude is a magnitude *of the same kind* as the original magnitudes.

This property includes the formation of multiples of a magnitude.

(3) If A and B be two magnitudes *of the same kind*, and A be greater than B, then another magnitude X *of the same kind* as A and B exists such that $B+X=A$.

(4) If A be any magnitude, and n any positive integer whatever, then a magnitude X *of the same kind* as A exists such that

$$nX=A.$$

(5) If A be greater than B, a multiple of B exists which is greater than A.

The fifth characteristic is known as the Axiom of Archimedes. It is not a consequence of the preceding four characteristics, but it is not proposed in

this address to enter into a discussion of any of the difficulties of non-Archimedean Geometry.

The following deduction from (1), (2), and (5) is specially useful in the Theory of Proportion:

If A and B are two magnitudes *of the same kind*, and any multiple whatever of A, say rA, is chosen, and any multiple whatever of B, say sB, is chosen, then one and only one of the alternatives

$$rA > sB, \quad rA = sB, \quad rA < sB$$

always exists, and it is assumed to be possible to determine which one of these alternatives exists.

10. In the second place, the fourth definition taken together with the third says that if A and B are two magnitudes *of the same kind*, then they also determine something else which Euclid calls a ratio, and which we regard as a number. It is denoted by $A : B$ or $B : A$, according to the order in which the magnitudes are taken.

The definition does not say what a ratio is, or how it is to be determined, *and this omission is one of the great difficulties* which present themselves to those who try to understand the argument.

It will be shown presently how the position of a ratio in, or with regard to, the system of rational numbers can be determined. Whenever this can be done the ratio is regarded as being itself a known number, *i.e.* it is known in the sense that all its properties can be determined.

Euclid's Point of View.

11. But before explaining how this is done, it is advisable to try to get at Euclid's point of view, so far as this is ascertainable; for though it is perhaps only of antiquarian interest, yet it forces itself upon one. *Now all the propositions that Euclid proves about ratios can be proved without introducing the idea of ratio at all.* When he states as a proposition that the ratio of A to B is the same as that of C to D, what he actually does in proving it is to show that the multiples of A are distributed amongst the multiples of B in the same way as the multiples of C are distributed amongst those of D, *e.g.* if pA lies between qB and $(q+1)B$, then he proves that pC lies between qD and $(q+1)D$; but if $pA = tB$, then he proves that $pC = tD$. Consequently the argument, with an important and interesting exception that will be noticed below (Art. 12(e)), deals only with *the multiples of the four magnitudes A, B, C, D and not in fact with their ratios.* De Morgan illustrated this by inventing his Theory of Relative Multiple Scales in his *Theory of the Connexion of Number and Magnitude.**

12. What then did Euclid understand by a ratio? In particular, did he or did he not regard it as a number? To this question there is in his work no clear or unambiguous answer. I can only set forth the evidence and leave you to judge of the issue.

On the one hand

(a) If he regarded a ratio as a magnitude, why does he give a demonstration of Prop. 11? Simson says "The words greater, the same or equal, lesser have a quite different meaning when applied to magnitudes and ratios, as is plain from the 5th and 7th definitions of Book V."

"That those things which are equal to the same are equal to one another is a most evident axiom when understood of magnitudes, yet Euclid does not make use of it or infer that those ratios which are the same to the same

*To this book I owe a very great debt, as it set me upon the track which I have since followed. I have developed and made use of the idea of Relative Multiple Scales in the first edition of my *Contents of the Fifth and Sixth Books of Euclid* and in the papers in Vols. XVI. and XIX. of the *Cambridge Philosophical Transactions*, but I abandoned it in the second edition for what seemed to me a simpler treatment.

ratio, are the same to one another: but explicitly demonstrates this in Prop. 11 of Book V."

(*b*) If he regarded a ratio as a magnitude, why does he give a demonstration of Prop. 13, which on the hypothesis that ratios are magnitudes amounts only to this:

If each of the symbols A, B, C represents a ratio, and if $A=B$, and $B>C$, then is $A>C$.

On the other hand, if he did not regard a ratio as a magnitude,

(*c*) Why does he speak of one ratio being greater than another in the Seventh Definition? The term greater, if used in the ordinary sense, can refer only to magnitudes.

(*d*) Why does he not supply a demonstration in connection with the Seventh Definition showing that if the four magnitudes A, B, C, D are such that integers r, s exist, such that

$$rA>sB, \text{ but } rC \not> sD, \qquad \text{(I)}$$

which are the conditions that $A:B>C:D$; then no integers r', s' can exist, such that

$$r'A<s'B, \text{ but } r'C \not< s'D, \qquad \text{(II)}$$

which are the conditions that $A:B<C:D$?*

Such a demonstration is unnecessary if a ratio is a magnitude, for $A:B>C:D$ and $A:B<C:D$ are inconsistent if $A:B$ and $C:D$ are magnitudes. On the other hand, if it is only a question of a comparison of the distribution of the multiples of A amongst those of B with the distribution of the multiples of C amongst those of D, then a demonstration of the incompatibility of the conditions (I) with the conditions (II) is essential.

(*e*) Why should he say in the proof of Prop. 10 that the statements

$$A:C=B:C \quad \text{and} \quad A:C<B:C$$

are inconsistent with $\quad A:C>B:C$?

For if ratios are magnitudes, this needs no proof; but if they are not so regarded, then the proof given by Euclid of this proposition does not follow from his definitions.

13. As to this matter, modern writers are equally in conflict. On the one hand, Stolz in his *Vorlesungen über allgemeine Arithmetik* (Erster Theil, p. 94) says: "In den auf uns gekommenen geometrischen Schriften des Alterthumes findet sich keine deutliche Spur der Ansicht, dass das Verhältniss zweier incommensurabelen Grössen eine Zahl sei."

Whilst, on the other hand, Max Simon (*Euklid und die sechs planimetrischen Bücher*), so far from agreeing in the usual view that the Greeks saw in the irrational no number, thinks it clear from Euclid, Book V., that they possessed a notion of number in all its generality.

The Arithmeticians and Algebraists of the Middle Ages called the ratios of incommensurable magnitudes "Numeri ficti" or "Numeri surdi," and regarded them as a necessary evil which had to be endured. Michael Stifel in his *Arithmetica Integra*, published in 1544, treated them as real numbers. His words amount to an assertion that each irrational number as well as each rational number has a single definite place in the ordered number series.†

The modern view of ratio is that it is in all cases a number. It cannot therefore be completely explained until irrational numbers have been defined. Commencing with positive integers, definitions are given of rational

* For a proof of this, see Heath, *l.c.* Vol. II. p. 130.

† (*Encyklopädie der mathematischen Wissenschaften*, Vol. I. A. 3, p. 51.)

fractions, negative integers and negative fractions. These constitute together the series of rational numbers, and rules are given for arranging them in a definite order.

Next suppose that some rule is given by means of which all the rational numbers can be separated into two classes, such that

(*a*) every number in the one class comes before every number in the other class,

(*b*) the first class has no last number,

(*c*) the second class has no first number,

then between the numbers in the two classes there is a gap.

This gap is filled by the creation of a number. It cannot be a rational number, because every rational number by hypothesis falls into one of the two classes. Consequently it is called an irrational number.

The irrational number appears therefore as the result of an extension of the idea of number. This prepares the way for the statement of the modern view of ratio, which is set out in Art. 16 below.

THE FIFTH DEFINITION.

14. This definition, though it does not define ratio, makes it possible to decide the extremely important question whether two ratios are equal, whether they be ratios of commensurable or incommensurable magnitudes.

This was sufficient for Euclid's purposes. There is a feature of this definition, which, so far as I know, was first noticed by Stolz.*

It is this:

That of the three sets of conditions,

(i) If $rA > sB$, then $rC > sD$;

(ii) If $rA = sB$, then $rC = sD$;

(iii) If $rA < sB$, then $rC < sD$,

where r, s are any positive integers whatever, the second is involved in the first and third, and is therefore superfluous. Proofs will be found in Stolz, *l.c.*, and in the second edition of my *Euclid V. and VI.*, p. 29, Art. 48.

It follows that in using the Fifth Definition it is sufficient to consider only the first and third sets of conditions, whilst the second may be omitted from consideration altogether. For a fuller discussion of this definition see the *Cambridge Philosophical Transactions*, Vol. XIX. p. 159.

THE SEVENTH DEFINITION.

15. This definition has a very intimate relation to Props. 8, 9 and 10, whose proofs depend on it. One point regarding it has already been noticed (Art. 12(*d*) above).

The first parts of Props. 7, 8, 9 and 10 form an important group. They are as follows:

Prop. 7 (i) If $A = B$, then $A : C = B : C$.

Prop. 9 (i) If $A : C = B : C$, then $A = B$.

Prop. 8 (i) If $A > B$, then $A : C > B : C$.

Prop. 10 (i) If $A : C > B : C$, then $A > B$.

Props. 7 and 8 together include the following important results, the last of which is included in the first:

(i) If $A > B$, then $A : C > B : C$;

(ii) If $A = B$, then $A : C = B : C$;

(iii) If $A < B$, then $A : C < B : C$.

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* *Vorlesungen über allgemeine Arithmetik*, Theil I. p. 87 (1885).

In these three results we touch what I believe to be the bed-rock of Euclid's ideas. Euclid indeed states them as propositions, and deduces them as consequences of his 5th and 7th definitions. But if we try to think about them and grasp their meaning, is it not clear that it is easier to see into their meaning than into that of the 5th and 7th definitions?

Suppose A, B and C represent segments of straight lines, and that A has the same length as B, then the relative magnitude of A when compared with C is the same as that of B when compared with C. And if A be a longer line than B, then the relative magnitude of A when compared with C is greater than the relative magnitude of B when compared with C.

(*To be continued.*)

THE POWER-SUM FORMULA AND THE BERNOULLIAN FUNCTION.

(In reference to A. C. Dixon's note on $1^m+2^m+3^m+\ldots+n^m$, on pp. 283-4 of this volume, the following further notes may be of interest. They are taken, with some alterations, from a paper communicated to the London Mathematical Society in February, 1910.)

I. The Power-Sum Formula.

1. Writing $$Sn^r \equiv 1^r+2^r+3^r+\ldots+n^r,$$

the ordinary formula may be written either

$$Sn^r = 1/(r+1).\{n^{r+1}+\tfrac{1}{2}(r+1,\ 1)n^r+B_1(r+1,\ 2)n^{r-1}$$
$$-B_2(r+1,\ 4)n^{r-3}+B_3(r+1,\ 6)n^{r-5}-\ldots\}, \quad \ldots\ldots\ldots\ldots(1)$$

or $$S(n-1)^r = 1/(r+1).\{n^{r+1}-\tfrac{1}{2}(r+1,\ 1)n^r+B_1(r+1,\ 2)n^{r-1}-\ldots\}, \quad \ldots(1\text{A})$$

where $(s,\ t)$ denotes $s!/\{t!(s-t)!\}$, and B_1, B_2, $B_3\ldots$ are Bernoulli's numbers; the series ending with the term in n^2 or in n according as r is odd or even. The expression in $\{\ \}$ presents two anomalies, viz. (i) the absence of a last term $(-)^{s-1}B_s$ for the case of $r=2s-1$, and (ii) the existence of the term in n^r, which spoils the regular progression of the series. The progression is otherwise quite regular if we introduce a Bernoulli's number $B_0 \equiv -1$.

2. The first anomaly is explained by the fact that the sum is taken between limits, *i.e.* if $\Sigma n^r \equiv \ldots+1^r+2^r+\ldots+n^r$, then $Sn^r=\Sigma n^r-\Sigma 0^r$, so that, since Sn^r is equal to a polynomial in n, this polynomial contains $n-0$ as a factor. But it will be seen later that the complete series, which, when $r \equiv 2s-1$, is equal to the curious expression

$$1^{2s-1}+2^{2s-1}+3^{2s-1}+\ldots+n^{2s-1}+(-)^{s+1}B_s/(2s),$$

is of importance for certain purposes.

3. The reason of the second anomaly is that the formula we are considering is essentially a formula, not for $1^r+2^r+\ldots+n^r$, or for $0^r+1^r+2^r+\ldots+n^r$, but for $\tfrac{1}{2}.0^r+1^r+2^r+\ldots+(n-1)^r+\tfrac{1}{2}n^r$; just as the occurrence of $\sqrt{n}$ in the approximate formula for $n!$ (when n is large) is due to the fact that we are essentially using a formula for $\tfrac{1}{2}\log 1+\log 2+\log 3+\ldots+\log(n-1)+\tfrac{1}{2}\log n$. The expression $\tfrac{1}{2}.0^r+1^r+2^r+\ldots+(n-1)^r+\tfrac{1}{2}n^r$ is the ordinary trapezoidal approximation to the area, between limits $x=0$ and $x=n$, of a figure whose ordinate is x^r; and the series composed of the terms involving B_1, B_2, ... in (1) gives the difference between this approximate value and the true value $(n^{r+1}-0^{r+1})/(r+1)$. We should naturally expect that this difference would be represented by a consistent expression in terms of n; and the addition of $\tfrac{1}{2}n^r$ to both sides produces an inconsistency.

4. To obtain a consistent expression for

$$1^r+2^r+\ldots+n^r \quad \text{or} \quad 0^r+1^r+2^r+\ldots+n^r$$

as a single entity, we observe that it is an approximate expression for the area of a figure whose ordinate, as before, is x^r, but which extends from $x=\frac{1}{2}$ or $x=-\frac{1}{2}$ to $x=n+\frac{1}{2}$. It is therefore a function of $n+\frac{1}{2}$ rather than of n. The following are particular cases, writing $m\equiv 2n+1$.

$$\begin{aligned}
Sn &=\tfrac{1}{2}(\tfrac{1}{2})^2(m^2-1),\\
Sn^2 &=\tfrac{1}{3}(\tfrac{1}{2})^3 m(m^2-1),\\
Sn^3 &=\tfrac{1}{4}(\tfrac{1}{2})^4(m^2-1)^2,\\
Sn^4 &=\tfrac{1}{15}(\tfrac{1}{2})^5 m(m^2-1)(3m^2-7),\\
Sn^5 &=\tfrac{1}{6}(\tfrac{1}{2})^6(m^2-1)^2(m^2-3),\\
Sn^6 &=\tfrac{1}{21}(\tfrac{1}{2})^7 m(m^2-1)(3m^4-18m^2+31),\\
Sn^7 &=\tfrac{1}{24}(\tfrac{1}{2})^8(m^2-1)^2(3m^4-22m^2+51),\\
Sn^8 &=\tfrac{1}{45}(\tfrac{1}{2})^9 m(m^2-1)(5m^6-55m^4+239m^2-381),\\
Sn^9 &=\tfrac{1}{10}(\tfrac{1}{2})^{10}(m^2-1)^2(m^2-5)(m^4-8m^2+31),\\
Sn^{10} &=\tfrac{1}{33}(\tfrac{1}{2})^{11} m(m^2-1)(m^2-5)(3m^6-37m^4+225m^2-511),\\
Sn^{11} &=\tfrac{1}{12}(\tfrac{1}{2})^{12}(m^2-1)^2(m^8-20m^6+190m^4-964m^2+2073).
\end{aligned}$$

5. The general formulæ for Sn^r in terms of n and in terms of $\nu\equiv n+\frac{1}{2}$ are most simply obtained by using the central-difference notation. Writing

$$\delta f(x)\equiv f(x+\tfrac{1}{2}h)-f(x-\tfrac{1}{2}h), \qquad \mu f(x)\equiv \tfrac{1}{2}\{f(x+\tfrac{1}{2}h)+f(x-\tfrac{1}{2}h)\},$$

$$\sigma f(x)\equiv \ldots+f(x-\tfrac{3}{2}h)+f(x-\tfrac{1}{2}h),$$

so that $\delta\sigma f(x)=f(x)$, we have $\delta=2\sinh(\frac{1}{2}hD)$, $\mu=\cosh(\frac{1}{2}hD)$, where D is the operator which converts a polynomial in x into its first derivative; and therefore, taking $h=1$, $x=n$,

$$\begin{aligned}
\ldots+(n-1)^r+\tfrac{1}{2}n^r &=\mu\sigma\,.\,n^r=\mu\delta^{-1}.\,n^r=\tfrac{1}{2}\coth\tfrac{1}{2}D\,.\,n^r\\
&=D^{-1}(1+B_1D^2/2!-B^2D^4/4!+B_3D^6/6!-\ldots)n^r \qquad \text{(2)}\\
&=\frac{n^{r+1}}{r+1}+B_1\frac{r}{2!}n^{r-1}-B_2\frac{r(r-1)(r-2)}{4!}n^{r-3}+\ldots, \qquad \text{(2A)}
\end{aligned}$$

and

$$\begin{aligned}
\ldots+(n-1)^r+n^r &=\sigma\,.\,\nu^r=\delta^{-1}.\,\nu^r=\tfrac{1}{2}\operatorname{cosech}\tfrac{1}{2}D\,.\,\nu^r\\
&=D^{-1}(1-p_1D^2/2!+p_2D^4/4!-p_3D^6/6!+\ldots)\nu^r \qquad \text{(3)}\\
&=\frac{1}{r+1}\{\nu^{r+1}-p_1(r+1,2)\nu^{r-1}+p_2(r+1,4)\nu^{r-3}-\ldots\} \qquad \text{(3A)}\\
&=\frac{1}{r+1}(\tfrac{1}{2})^{r+1}\{m^{r+1}-P_1(r+1,2)m^{r-1}+P_2(r+1,4)m^{r-3}-\ldots\}, \qquad \text{(3B)}
\end{aligned}$$

where $\quad P_q\equiv(2^{2q}-2)B_q,\quad p_q=(\frac{1}{2})^{2q}P_q=\{1-(\frac{1}{2})^{2q-1}\}B_q.$ ....(4)

6. For $\frac{1}{2}.0^r+1^r+2^r+\ldots+(n-1)^r+\frac{1}{2}n^r$, the series in (2A) will (§2) end with the term in n^2 or in n. To find the last term of the series in (3B), as a formula for Sn^r, we may proceed as follows.

(i) The term is found by taking the series between limits $\nu=\frac{1}{2}$ or $\nu=-\frac{1}{2}$ and $\nu=\nu$; *i.e.* it is found by the condition that the series in (3B) has m^2-1 as a factor. We have therefore to find the value of

$$1-P_1(r+1,\,2)+P_2(r+1,\,4)-\ldots,$$

continued up to the term in $(r+1,\,r-1)$ or in $(r+1,\,r)$ according as r is odd or even.

(ii) This involves equating coefficients in

$$\{1+\theta+\theta^2/2!+\theta^3/3!+\dots\}\{1-P_1\theta^2/2!+P_2\theta^4/4!-P_3\theta^6/6!+\dots\}$$
$$=e^\theta.\,\theta\operatorname{cosech}\theta=\theta+\theta\coth\theta$$
$$=1+\theta+2^2B_1\theta^2/2!-2^4B_2\theta^4/4!+2^6B_3\theta^6/6!-\dots. \qquad(5$$

(a) Let $r=2s-1$. Then, equating coefficients of $\theta^{2s}/(2s)!$ in (5),

$$1-P_1(2s,2)+P_2(2s,4)-\dots+(-)^sP_s=(-)^{s+1}2^{2s}B_s. \qquad(6)$$

Hence, if we write

$$\lambda_q\equiv P_q+2^{2q}B_q=2(P_q+B_q)=2(2^{2q}-1)B_q, \qquad(7)$$

we have

$$1-P_1(2s,2)+P_2(2s,4)-\dots+(-)^{s-1}P_{s-1}(2s,2s-2)+(-)^s\lambda_s=0,$$

and therefore

$$Sn^{2s-1}=\frac{1}{2s}(\tfrac{1}{2})^{2s}\{m^{2s}-P_1(2s,2)m^{2s-2}+P_2(2s,4)m^{2s-4}-\dots$$
$$+(-)^{s-1}P_{s-1}(2s,2s-2)m^2+(-)^s\lambda_s\} \qquad(8)$$
$$=\frac{1}{2s}\{\nu^{2s}-p_1(2s,2)\nu^{2s-2}+p_2(2s,4)\nu^{2s-4}-\dots$$
$$+(-)^{s-1}p_{s-1}(2s,2s-2)\nu^2+(-)^s\lambda_s/2^{2s}\}. \qquad(8\text{A})$$

(b) Let $r=2s$. Then, equating coefficients of $\theta^{2s+1}/(2s+1)$ in (5),

$$1-P_1(2s+1,2)+P_2(2s+1,4)-\dots+(-)^sP_s(2s+1,2s)=0; \qquad(9)$$

and therefore

$$Sn^{2s}=\frac{1}{2s+1}(\tfrac{1}{2})^{2s+1}\{m^{2s+1}-P_1(2s+1,2)m^{2s-1}+P_2(2s+1,4)m^{2s-3}-\dots$$
$$+(-)^sP_s(2s+1,2s)m\} \qquad(10)$$
$$=\frac{1}{2s+1}\{\nu^{2s+1}-p_1(2s+1,2)\nu^{2s-1}+p_2(2s+1,4)\nu^{2s-3}-\dots$$
$$+(-)^sp_s(2s+1,2s)\nu\}. \qquad(10\text{A})$$

(iii) We see that, if r is even, the expression for Sn^r is divisible by m, *i.e.* by $2n+1$; and that, whether r is even or odd, it is divisible by m^2-1, and therefore by $n(n+1)$.

7. Although the expression for Σn^r in terms of $n+\frac{1}{2}$ or $2n+1$ is simpler algebraically than the expression in terms of n, it is not so simple arithmetically, since the coefficients are unmanageably large. We know that $2B_q$ (except for $q=0$) is a fraction whose numerator and denominator are odd; and it may be shown that $\lambda_q\equiv 2B_q+2P_q$ is an odd integer. It follows from (4) that P_q is obtained from B_q by dividing its denominator by 2 and multiplying its numerator by $2^{2q-1}-1$; and this multiplier soon becomes large. The following are the values of λ_q and of P_q up to $q=10$.

$$\lambda_0=0,\ \lambda_1=1,\ \lambda_2=1,\ \lambda_3=3,\ \lambda_4=17,\ \lambda_5=155,\ \lambda_6=2073,\ \lambda_7=38227,$$
$$\lambda_8=929569,\ \lambda_9=28820619,\ \lambda_{10}=1109652905;$$
$$P_0=1,\ P_1=1/3,\ P_2=7/15,\ P_3=31/21,\ P_4=127/15,\ P_5=2555/33,$$
$$P_6=1414477/1365,\ P_7=57337/3,\ P_8=118518239/255,$$
$$P_9=5749691557/399,\ P_{10}=91546277357/165.$$

II. The Bernoullian Function.

8. The Bernoullian function of degree k is usually defined as being

$$\phi_k(x)\equiv x^k-\tfrac{1}{2}x^{k-1}+B_1(k,2)x^{k-2}-B_2(k,4)x^{k-4}+B_3(k,6)x^{k-6}-\dots; \qquad(11)$$

the series ending with the term in x^2 or in x according as k is even or odd. This is for $k>1$; for $k=1$, $\phi_1(x)$ is usually taken to be x, but it would be

more consistent to take it to be $x-\frac{1}{2}$. Various properties of the function are given by Bromwich, *Theory of Infinite Series*, pp. 235-7. (In the last line on p. 237, $(-)^{m-1}$ should be $(-)^m$.)

9. The occurrence of $(-)^s\lambda_s$, in place of $(-)^sP_s$, in the last term of the series in (8), is due to the absence of the term in n^0 in the formula for Sn^{2s-1} in terms of n; and we should obtain neater results by introducing this latter term. We therefore write

$$\Phi_k(x) \equiv x^k - \tfrac{1}{2}kx^{k-1} + B_1(k, 2)x^{k-2} - B_2(k, 4)x^{k-4} + B_3(k, 6)x^{k-6} - \dots, \quad \text{(12)}$$

$$\Psi_k(x) \equiv x^k - p_1(k, 2)x^{k-2} + p_2(k, 4)x^{k-4} - p_3(k, 6)x^{k-6} + \dots; \quad \text{(13)}$$

the series in each case continuing until the coefficients become zero. Then we shall have

$$\Psi_k(x) = \Phi_k(x+\tfrac{1}{2}), \quad \text{(14)}$$

$$e^{xt}(\tfrac{1}{2}t\coth\tfrac{1}{2}t - \tfrac{1}{2}t) = \sum_{k=0}^{k=\infty} \Phi_k(x) \,.\, t^k/k!, \quad \text{(15)}$$

$$e^{xt} \,.\, \tfrac{1}{2}t \operatorname{cosech}\tfrac{1}{2}t = \sum_{k=0}^{k=\infty} \Psi_k(x) \,.\, t^k/k!. \quad \text{(16)}$$

For $k=0$ and $k=1$ the functions will be

$$\Phi_0(x)=1, \quad \Psi_0(x)=1, \quad \Phi_1(x)=x-\tfrac{1}{2}, \quad \Psi_1(x)=x.$$

It will be found that the following relations hold.

(i) $\Phi_k(x+1) - \Phi_k(x) = \Psi_k(x+\frac{1}{2}) - \Psi_k(x-\frac{1}{2}) = kx^{k-1}$.

(ii) $\Phi_k(1-x) = (-)^k\Phi(x)$; $\Psi_k(-x) = (-)^k\Psi_k(x)$.

(iii) $\Phi_k'(x) = k\Phi_{k-1}(x)$; $\Psi_k'(x) = k\Psi_{k-1}(x)$.

(iv) If $q>0$, $\Phi_{2q+1}(x)$ contains $x(x-\frac{1}{2})(x-1)$ as a factor, and $\Psi_{2q+1}(x)$ contains $x(x^2-\frac{1}{4})$ as a factor.

(v) If $q>1$, $\Phi_{2q}(x)+(-)^qB_q$ contains $x^2(x-1)^2$ as a factor, and $\Psi_{2q}(x)+(-)^qB_q$ contains $(x^2-\frac{1}{4})^2$ as a factor.

(vi) If $q>0$,

$$\Phi_{2q+1}(0) = \Phi_{2q+1}(\tfrac{1}{2}) = \Phi_{2q+1}(1) = \Psi_{2q+1}(\pm\tfrac{1}{2}) = \Psi_{2q+1}(0) = 0.$$

(vii) $\left.\begin{aligned} &\Phi_{2q}(0) = \Phi_{2q}(1) = \Psi_{2q}(\pm\tfrac{1}{2}) = (-)^{q-1}B_q; \\ &\Phi_{2q}(\tfrac{1}{2}) = \Psi_{2q}(0) = (-)^q p_q. \end{aligned}\right\}$

(viii) $\Phi_{2q+1}(x)$ is of sign $(-)^{q+1}$ from $x=0$ to $x=\frac{1}{2}$, and of sign $(-)^q$ from $x=\frac{1}{2}$ to $x=1$, and has no zero value between $x=0$ and $x=1$ except at $x=\frac{1}{2}$; and $\Psi_{2q+1}(x)$ is of sign $(-)^{q+1}$ from $x=-\frac{1}{2}$ to $x=0$, and of sign $(-)^q$ from $x=0$ to $x=\frac{1}{2}$, and has no zero value between $x=-\frac{1}{2}$ and $x=\frac{1}{2}$ except at $x=0$.

(ix) $\Phi_{2q}(x)$ (for $q>0$) is zero once only between $x=0$ and $x=\frac{1}{2}$, and once only between $x=\frac{1}{2}$ and $x=1$, and has no stationary value between $x=0$ and $x=1$ except at $x=\frac{1}{2}$; and $\Psi_{2q}(x)$ (for $q>0$) is zero once only between $x=-\frac{1}{2}$ and $x=0$, and once only between $x=0$ and $x=\frac{1}{2}$, and has no stationary value between $x=-\frac{1}{2}$ and $x=\frac{1}{2}$ except at $x=0$.

(x) $\Phi_{2q}(x)+(-)^qB_q(=\Phi_{2q}(x)-\Phi_{2q}(0)=\Phi_{2q}(x)-\Phi_{2q}(1))$ is of sign $(-)^q$ from $x=0$ to $x=1$; and $\Psi_{2q}(x)+(-)^qB_q(=\Psi_{2q}(x)-\Psi_{2q}(\pm\frac{1}{2}))$ is of sign $(-)^q$ from $x=-\frac{1}{2}$ to $x=\frac{1}{2}$. Also $\Phi_{2q}(x)+(-)^{q-1}p_q(=\Phi_{2q}(x)-\Phi_{2q}(\frac{1}{2}))$ is of sign $(-)^{q-1}$ from $x=0$ to $x=\frac{1}{2}$, and from $x=\frac{1}{2}$ to $x=1$, and $\Psi_{2q}(x)+(-)^{q-1}p_q$ $(=\Psi_{2q}(x)-\Psi_{2q}(0))$ is of sign $(-)^{q-1}$ from $x=-\frac{1}{2}$ to $x=0$, and from $x=0$ to $x=\frac{1}{2}$.

(xi) If (so as to include the case of $r=0$) we define Sn^r as being $\frac{1}{2}.0^r+1^r+2^r+\ldots+n^r$, then

$$\begin{aligned} Sn^{2q-1} &= 1/(2q).\{\Phi_{2q}(n+1)+(-)^q B_q\} \\ &= 1/(2q).\{\Psi_{2q}(n+\tfrac{1}{2})+(-)^q B_q\}, \\ Sn^{2q} &= 1/(2q+1).\Phi_{2q+1}(n+1) \\ &= 1/(2q+1).\Psi_{2q+1}(n+\tfrac{1}{2}). \end{aligned}$$

W. F. SHEPPARD.

(*To be continued.*)

NOTICE.

WE have to thank the Officers of the Congress whose photographs are reproduced in this volume for their kind consent in allowing them to appear.

We are also indebted to Mr. John Murray for the electro of Roubiliac's statue of Newton. To Messrs. J. Palmer Clarke, Cambridge, Lafayette & Co., Ltd., London, and Elliott & Fry, Ltd., London, we beg to express our sense of the courtesy with which they accorded the permission it was necessary to obtain.

LOCAL BRANCHES.

THE N. WALES BRANCH.

THE N. Wales branch of the Mathematical Association recently organised a conference of mathematical teachers from the primary schools, secondary schools, training colleges, and the University College of N. Wales, which was held at Bangor on May 9. Prof. G. H. Bryan took the chair, and Principal Sir H. Reichel also attended.

The Chairman said that it was very important that such conferences should be held at the present time on account of the great changes which had recently taken place in the teaching of children and in the treatment of the different subjects. In one branch, indeed, teachers had continually to wage war against what he had previously characterised as "England's neglect of Mathematics." In some quarters it did not seem to be realised what an important part the teachers of the exact science played in our social and economic life. Statistics and political questions brought them face to face with a big problem, and the introduction of such questions would largely obtain in the schools of the future, inasmuch as in time to come political and economic problems would be resolved into questions of statistics. A comparison of the methods of teaching mathematics adopted now and twenty years ago would reveal the great improvement which had taken place. In regard to geometry, formerly the idea seemed to consist in the failing of the pupils to do riders, and in the learning by rote of theorems which were promptly forgotten when the examination day came round or when the distinguishing letters of the diagram were changed. A mistaken idea was abroad that mathematicians of high order made bad teachers. How absurd this was could be seen from the fact that Smith's prizemen and the products of Part II. have applied the same methods of research used in original papers to subjects taught out of elementary text-books. These men had cast a powerful searchlight on obscure and difficult parts of our elementary curricula, and had asked what was the good of pupils' studying things beyond their grasp. As a result, modern teaching tended more and more to give pupils a clear understanding of quantity and space instead of the mere power of juggling with collections of letters and of old and obscure systems. Nevertheless, it was not a destructive policy of teaching that they were inquiring after, but a rational one. Continuity of purpose did not imply uniformity of methods.

There was probably no class of individuals who were more enthusiastic, energetic, and untiring in their efforts to increase the national efficiency than mathematical teachers, however little this might be realised. The speaker himself had been much impressed at recent conferences by papers read by mathematical teachers, particularly of the elementary and secondary schools, which had shown exceptional insight into and careful forethought given to their work.

[Then followed papers by Mr. R. W. Jones and Mr. W. J. Walker, which will later appear in the *Gazette.*]

THE SYDNEY BRANCH.

The half-yearly meeting of this Branch was held at Sydney on May 24th. Mr. Elliot, Government Inspector of Secondary Schools, read a paper on the teaching of mathematics in Secondary Schools, which was followed by a discussion.

MATHEMATICAL NOTES.

372. [V. a ; D. 6. b.] *Note on the Logarithmic Series.*

The proof usually given for the Logarithmic Series depends upon the identification of exponential and binomial expansions of the expression $(1+x)^y$, and involves the rearrangement of a double series (see for example Chrystal's *Algebra*, vol. ii. p. 238). At the stage when the logarithmic theory is introduced, a student has not studied double series, and the question of rearrangement is not gone into (C. Smith's *Algebra*, p. 381). The proof, which in any case is a little artificial, thereby becomes unsound.

This note is written to suggest a different proof for the logarithmic series ; and at the same time the steps of the theory are summarized.

1°. Proof of the exponential theorem by Prof. Hill's method (C. Smith's *Algebra*, p. 373).

$$e^x \text{ is the sum of the series } 1+x+\frac{x^2}{2!}+\ldots+\frac{x^n}{n!}+\ldots \qquad (1)$$

2°. Compare the series (1) with $1+x+\dfrac{x^2}{2!}\left(1+\dfrac{x}{3}+\dfrac{x^2}{3^2}+\dfrac{x^3}{3^3}+\ldots\right)$.

We have then
$$e^x=1+x+\frac{x^2}{2!}R,$$
where $|R|<\dfrac{1}{1-\dfrac{|x|}{3}}$ for $|x|<3$; so that $|R|<\dfrac{3}{2}$ for $|x|<1$.

3°. Hence
$$(e^x-1)/x=1+\frac{x}{2!}R,$$
and therefore
$$\operatorname*{Lt}_{x\to 0}(e^x-1)/x=1. \qquad (2)$$

4°. Writing $a=e^{\log a}$,
$$\operatorname*{Lt}_{h\to 0}(a^h-1)/h=\log a. \qquad (3)$$

5°. (*Proof of Logarithmic Series.*) Whence $\log(1+x)=\mathrm{Lt}[(1+x)^h-1]/h$;

$$\therefore\ \log(1+x)=\operatorname*{Lt}_{h\to 0}\left\{x+(h-1)\frac{x^2}{2!}+(h-1)(h-2)\frac{x^3}{3!}+\ldots\right.$$
$$\left.+(h-1)(h-2)\ldots(h-r+1)\frac{x^r}{r!}R'\right\},$$

where $|R'|<$ the sum of the series $1+\left|\dfrac{h-r}{r+1}x\right|+\left|\dfrac{\overline{h-r}.\overline{h-r-1}}{\overline{r+1}.\overline{r+2}}x^2\right|+\ldots$ (4)

Take $0<h<1$, and compare (4) with $1+|x|+|x|^2+\ldots$.

Hence $$|R'|<\frac{1}{1-|x|}, \text{ if } |x|<1.$$

Hence $$\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots+(-1)^{r-1}\frac{x^r}{r!}R'_0,$$

where R'_0 is the limit of R', and is therefore $<1/(1-|x|)$.

[The existence of R'_0 is easily established: if $x<0$, R' increases as h diminishes: if $x>0$, R' is the difference of the sums of two series, which sums remain finite and increase as h diminishes.]

6°. The expression $$\underset{n\to\infty}{\mathrm{Lt}}\left(1+\frac{1}{n}\right)^{nx}=\underset{n\to\infty}{\mathrm{Lt}}\, e^{nx\log(1+1/n)}$$
$$=\underset{n\to\infty}{\mathrm{Lt}}\, e^{nx(1/n-1/2n^2R')}=\underset{n\to\infty}{\mathrm{Lt}}\, e^{x(1-1/2nR')},$$

where $|R'|<1\Big/\left(1-\dfrac{1}{n}\right)$; so that $|R'|<2$ for $n>2$.

Hence $$\underset{n\to\infty}{\mathrm{Lt}}\left(1+\frac{1}{n}\right)^{nx}=e^x.$$

H. Bryon Heywood.

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373. [I. 17.] *A few cases of factors for a sum of two squares.*

(i) $x^{12}+(4x^2y^4+2y^6)^2$
$$=(x^6+4x^5y+8x^4y^2+10x^3y^3+8x^2y^4+4xy^5+2y^6)$$
$$\times(x^6-4x^5y+8x^4y^2-10x^3y^3+8x^2y^4-4xy^5+2y^6).$$

If $x=1$ and $y=10$, we get
$$(2040000)^2+1=(2490841)\times(1670761).$$

(ii) $(x^6-8x^4y^2+8x^2y^4)^2+(4x^2y^4-2y^6)^2$
$$=(x^6+4x^5y-10x^3y^3+4xy^5+2y^6)$$
$$\times(x^6-4x^5y+10x^3y^3-4xy^5+2y^6).$$

(iii) $(8x^6+36x^4y^2)^2+(27x^2y^4+81y^6)^2$
$$=(8x^6+24x^5y+72x^4y^2+144x^3y^3+189x^2y^4+162xy^5+81y^6)$$
$$\times(8x^6-24x^5y+72x^4y^2-144x^3y^3+189x^2y^4-162xy^5+81y^6).$$

If $x=10$ and $y=1$, this gives
$$(8360000)^2+(2781)^2=11284601\times6193361.$$

(iv) $(2x^6-150x^4y^2)^2+(18x^2y^4+y^6)^2$
$$=(2x^6+36x^5y+174x^4y^2+110x^3y^3+36x^2y^4+6xy^5+y^6)$$
$$\times(2x^6-36x^5y+174x^4y^2-110x^3y^3+36x^2y^4-6xy^5+y^6).$$

(v) $(x^6+8x^4y^2)^2+(18x^2y^4+81y^6)^2$
$$=(x^6+2x^5y+12x^4y^2+21x^3y^3+36x^2y^4+54xy^5+81y^6)$$
$$\times(x^6-2x^5y+12x^4y^2-21x^3y^3+36x^2y^4-54xy^5+81y^6).$$

A sum of three squares which has factors:
$$(2x^4)^2+(x^2y^2)^2+(2y^4)^2$$
$$=(2x^4+2x^3y+x^2y^2+2xy^3+2y^4)$$
$$\times(2x^4-2x^3y+x^2y^2-2xy^3+2y^4).$$

G. Osborn.

374. [A. 3. b.] *Any symmetric function of the roots of an equation is a function of the coefficients.*

(The following direct proof is shorter than the usual one depending on Newton's Theorem. The process occurs in an example, Burnside and Panton, p. 325, 3rd ed.)

Let the n roots be $a, \beta, \dots \nu$, and let the assigned function be $\Sigma a^a \beta^b \dots \lambda^l$, where each term involves p different roots, and the highest index of any one root is q.

If $q=1$, then Σ is equal to a coefficient; if $p=n$, then Σ is the product of the last coefficient and a symmetric function with a lower q. So the theorem will be proved if we can express any symmetric function in terms of others with either a lower q, or the same q and a higher p; for with each of the latter we can continue until $p=n$, and so obtain a lower q in every sum; and then we can repeat the process until $q=1$.

Now consider the product $\Sigma a\beta \dots \lambda . \Sigma a^{a-1}\beta^{b-1} \dots \lambda^{l-1}$, of which the first factor has $q=1$, and the second has a lower q than the assigned function. This product is equal to the sum of a set of symmetric functions, of which none has a higher q than the assigned function, and each has a higher p except one, and that one is the assigned function itself, which can therefore be expressed in the way described above. This proves the theorem, and Newton's as a particular case. H. P. HUDSON.

375. [I. 2. b.] May I make the following remarks apropos of Mr. Lupton's article "Furor Arithmeticus" (*Math. Gaz.*, May 1910, pp. 273 *et seq.*)?

(i) The value given for 40! is incorrect; this will be readily seen from the fact that the number is not a multiple of nine. M[lle] and MM. Chanzy, of Nancy, have found

$$40! = 815\,915\,283\,247\,897\,734\,345\,611\,269\,596 \dots .$$

(ii) The largest known primes,

$$5 . 2^{75}+1 = 188\,894\,659\,314\,785\,808\,547\,841,$$

which is a factor of Fermat's number F_{73}, or $2^{2^{73}}+1$ (*v.* Dr. J. C. Morehead's article in the *Bull. of the Amer. Math. Soc.* 1906, translated into French in *Sphinx Oedipe*, April 1911).

According to Mr. Powers, of Denver, Colorado, the number $2^{83}-1$ is a prime. This number has 27 digits (*Amer. Math. Monthly*, Nov. 1911).

Edouard Lucas proved that $2^{127}-1$ is a prime. This number has 39 digits (*Bull. du prince Boncompagni*, Rome, 1877).

(iii) I have a complete "historique" of π, giving 707 decimal places. If any reader of the *Gazette* would care to see them they are at his disposal.

A. GÉRARDIN.

376. [I. 2.] *The Introduction to the Idea of a Negative Number.*

In the review of Prof. Tannery's book in your December issue, it is mentioned that in leading up to the idea of a negative number, the author confines himself to concepts arising out of the idea of number, using the series:

$$\dots -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \dots .$$

It is difficult to believe that this series of numbers is intelligible to a *beginner*, unless made real and concrete by some such device as your reviewer suggests. I have frequently tested boys who have been taught the manipulation of negative numbers by the method suggested above (without the looking-glass device suggested by your reviewer): I have given the boys the series of numbers in any order and told them to arrange them in the right order in one series. The result is nearly always:

$$0 \quad -1 \quad -2 \quad -3 \quad 1 \quad 2 \quad 3.$$

This result is given by boys who are neither backward nor dull. Even the few who can arrange the numbers in the right order (and these are not always those who can manipulate negative numbers correctly) say that they do not understand the series, and that it is no help to them.

A boy who has worked at algebra in which $x-y$ is intelligible only when x is greater than y has no idea of negative numbers at all; his ideas of

"series" are very vague, the only series he understands is the series of natural (positive) numbers, 0 as a number in a series is unintelligible. In fact, the series of numbers is confusion worse confounded, and *begs the whole question*; it is only the boy who has grasped in some other way the ideas to which the series is supposed to lead who understands the series at all.

The looking-glass device suggested by your reviewer makes a direct appeal to the senses, but would be difficult to use before a class. Is any device better than the use of the thermometer? The actual instrument need not be used, in fact the diagram is better. It is easy to understand that the mercury may fall to zero and any number of degrees below, and the idea of negative numbers and their order in series follows at once.

St. Paul's School, West Kensington, W. — E. W. P. TEMPERLEY.

377. [X. 4. b.] *Graphs.*

In a letter which I had the privilege of reading to the London Branch, Professor Filon spoke of the "tedious and brutalising" arithmetic which accompanies graphs as their shadow.

This arithmetic is often quite unnecessary, as he proceeded to point out.

A good example came recently to my notice. Several teachers criticised a question, "Draw the graph of $2\sin\theta+\cos\theta$," as involving an excessive amount of computation.

May I suggest to them the following as a profitable exercise:

"Given a sheet of plain paper, a drawing board, two non-graduated set squares, a compass with a pencil point, and no other appliances, books or tables whatever; draw a graph of $2\sin\theta+\cos\theta$ to scale."

A variant of Note 310 was set at Dartmouth in December, 1909.

"Draw a graph of the straight line

$$5x+2y=3. \quad \text{(Unit 2 inches.)}$$

Draw a circle, centre the origin O, radius 2 inches, cutting the graph in P and Q and the axis Ox in A.

If $AOP=\theta$, express the x and y coordinates of P in terms of θ.

Deduce from the graph the solutions of the equation

$$5\cos\theta+2\sin\theta=3."$$

This graph is rather more difficult to draw, but the results are rather more obvious than in Note 310. — C. S. JACKSON.

378. [Q. b.] In the July number of the *Gazette* Professor Bryan gives an interesting extension of a paper of mine in the *Proc. Edin. Math. Soc.* containing an elementary interpretation of the Bolyai-Lobatschewsky *Geometry.* However, from his introductory remarks it might be inferred that the discovery of the system of ideal lines, etc., used in that paper had been made by me. As a matter of fact that system, or something akin to it, has been employed by various writers (Klein, Poincaré, Liebmann, and others). At the beginning of my paper I refer to Wellstein's discussion in the well-known Weber-Wellstein *Encyclopädie der Elementar Mathematik*; and I mention that my work was suggested by that discussion, and was to be regarded as an extension of it.

May I add that in Professor Bryan's description of the Euclidean case there is a slight error. The "ideal point" is a single point. In the Non-Euclidean case examined, it is a "pair of points." In other interpretations, which have been given, it has still less the character of indivisibility. [Cf. Weber-Wellstein, *loc. cit.* Bd. II. p. 52; Bonola, *La Geometria Non-Euclidea*, §72; Poincaré, *La Science et Hypothèse*, pp. 56-58.] — H. S. CARSLAW.

QUERIES.

(81) Wanted, a solution of the problem:—A drawer of depth d and breadth b has coefficient of friction μ along the sides, and is pulled out by an eccentric force perpendicular to the face; find the condition of jamming.

R. F. M.

(82) Can the theory of parallels be founded on the axiom that the length of a finite straight line remains constant when moved from one place to another? If so, where has it been worked out?

H. PIAGGIO.

ANSWERS TO QUERIES.

[**71, p. 330, vol. v.**] (Second Solution.) If $\phi(x)$ is a polynomial containing only odd powers of x, while $\psi(x)$ contains only even powers, and if

$$\phi(2n+1)+\psi(2n)=A_0+A_1n+A_2\frac{n(n-1)}{2!}+A_3\frac{n(n-1)(n-2)}{3!}+\dots,$$

then
$$A_0-\frac{A_1}{2}+\frac{A_2}{2^2}+\frac{A_3}{2^3}+\dots=\psi(0).$$

As the polynomials contain only a finite number of terms, the general theorem can be built up from particular cases.

Case (1). $\qquad \phi(2n+1)\equiv(2n+1)^{2p+1}.$

The coefficient A_0 is the leading term of the series formed by giving n the values 0, 1, 2 ..., and A_1, A_2 ... are the leading terms of successive orders of differences. Accordingly

$$A_r=(2r+1)^{2p+1}-{}_rC_1(2r-1)^{2p+1}+{}_rC_2(2r-3)^{2p+1}\dots+(-1)^r.\,1^{2p+1}$$

$$=\text{coefft. of } \frac{x^{2p+1}}{\underline{|2p+1}} \text{ in } e^{\overline{2r+1}x}-{}_rC_1e^{\overline{2r-1}x}+{}_rC_2e^{\overline{2r-3}x}-\dots\dots+(-1)^re^x$$

$$=\quad ,, \quad ,, \quad \text{in } e^x(e^{2x}-1)^r;$$

$$\therefore\ A_0-\frac{A_1}{2}+\frac{A_2}{2^2}-\frac{A_3}{2^3}+\dots$$

$$=\text{coefft. of } \frac{x^{2p+1}}{\underline{|2p+1}} \text{ in } e^x\left[1-\frac{e^{2x}-1}{2}+\left(\frac{e^{2x}-1}{2}\right)^2+\dots\right]$$

$$=\quad ,, \quad ,, \quad \text{in } e^x\left[1+\frac{e^{2x}-1}{2}\right]^{-1}$$

$$=\quad ,, \quad ,, \quad \text{in } \frac{2}{e^x+e^{-x}}$$

$$=\quad ,, \quad ,, \quad \text{in an even function of } x$$

$$=0.$$

Case (2). $\qquad \psi(2n)\equiv(2n)^{2p}.$

Here $\quad A_r=(2r)^{2p}-{}_rC_1(2r-2)^{2p}+{}_rC_2(2r-4)^{2p}\dots+(-1)^{r-1}{}_rC_{r-1}$

$$=\text{coefft. of } \frac{x^{2p}}{\underline{|2p}} \text{ in } e^{2rx}-{}_rC_1e^{(2r-2)x}+\dots+(-1)^{r-1}{}_rC_{r-1}e^{2x}$$

$$=\quad ,, \quad ,, \quad \text{in } (e^{2x}-1)^r-(-1)^r;$$

$$\therefore\ A_0 - \frac{A_1}{2} + \frac{A_2}{2^2} - \frac{A_3}{2_3} + \dots$$

$$\begin{aligned} &= \text{coefft. of } \frac{x^{2p}}{\lfloor 2p} \text{ in } \left[1 + \frac{e^{2x}-1}{2}\right]^{-1} - 2 \\ &= \quad ,, \quad ,, \text{ in } -1 - \frac{e^x - e^{-x}}{e^x + e^{-x}} \\ &= \quad ,, \quad ,, \text{ in an odd function of } x \\ &= 0. \end{aligned}$$

In each part of the proof x must be chosen so that $e^{2x} < 3$.

F. J. W. WHIPPLE.

[75, p. 159, vol. vi.] (Second Solution.) Proof: By similar triangles

$$\frac{XZ}{XA} = \frac{YZ}{YB} \quad \text{and} \quad \frac{XZ}{ZC} = \frac{XY}{YB};$$

$$\therefore\ \frac{XZ}{XA} + \frac{XZ}{ZC} = \frac{XY + YZ}{YB} = \frac{XZ}{YB};$$

$$\therefore\ XA^{-1} + ZC^{-1} = YB^{-1}.$$

The application of this relation to the determination of the joint resistance of several resistances in parallel was, I believe, first pointed out by the *Elektrotechnische Zeitschrift*, and it appeared later in *Le Génie Civil*, May 2, 1891, and in the *Electrician*, Dec. 18, 1891.

The following is a simple and obvious extension to any number of quantities, such as resistances in parallel, electrical capacities in series, focal lengths of thin lenses in contact, etc., connected by the relation

$$X^{-1} = \Sigma x^{-1}.$$

As apparently it is not given in the text-books it may be of some interest.

On any two parallel lines mark off lengths AB, CD, CE, CF, CG, etc., representing the quantities to scale, and combine in pairs as shown in the figure. The last parallel PT will represent the quantity sought. AB and CD are for convenience drawn perpendicular to AC. Directed quantities, such as focal lengths, must be drawn above or below AC according to sign. If $AB=3$, $CD=6$, $CE=5$, $CF=4$, $CG=-2$, we get $PT=2{\cdot}22$.

For the application to combinations of co-axial refracting surfaces, see paper by H. S. Allen, *Proceedings of the Physical Society*, vol. xxi. p. 480.

J. A. TOMKINS.

[80, p. 297, vol. vi.] Denote BAD $(=\pi/8)$ by α, and let $ABD = \alpha + \epsilon$.

By drawing perpendiculars from A and E on BC, we find

$$\tan(2\alpha + \epsilon) : \tan 2(\alpha + \epsilon) = 7 : 10;$$

$$\therefore\ \sin\epsilon : \cos 3\epsilon = 3 : 17, \text{ since } 4\alpha = \pi/2.$$

Thus $\epsilon = 9^\circ$ approximately,

and $$BC = 20 . \frac{\sin 54^\circ}{\sin 63^\circ} = 18{\cdot}2 \text{ approximately.}$$

R. F. DAVIS.

THE PILLORY.

"A cyclist finds that when he makes his best speed the wind appears to be always within 30° of right ahead. Explain this; and if the wind blows up to 8 miles an hour find the cyclist's greatest speed."

University of Edinburgh.

It is conjectured that the meaning intended was this. Whatever the direction or velocity of the wind, supposed to remain constant, the cyclist can

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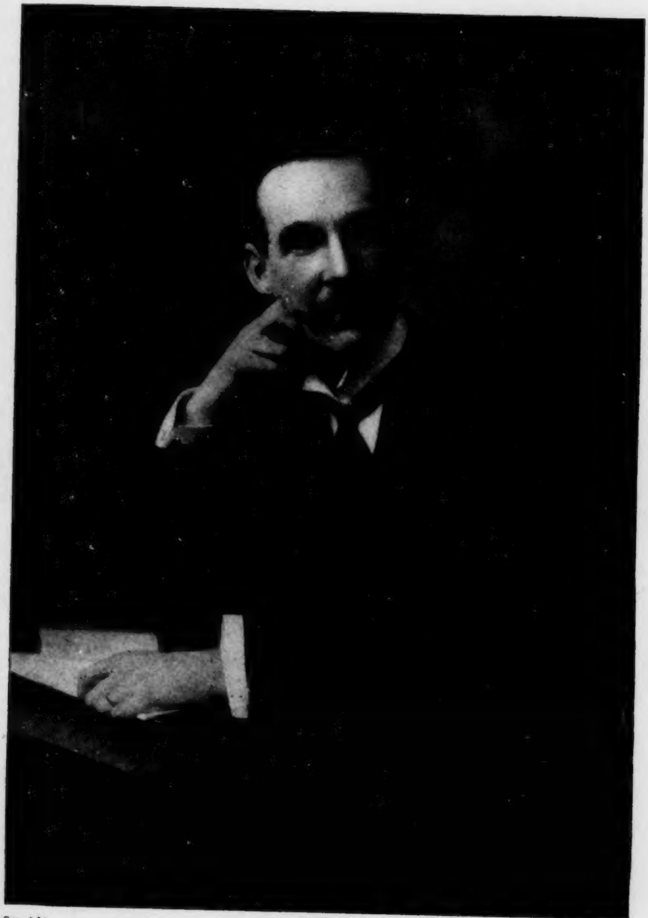

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PROFESSOR E. W. HOBSON, F.R.S.

Facing p. 342.

ride fast enough *in a straight line*, to bring the apparent wind within 30° of right ahead. It is easily seen that the most unfavourable case is when the true direction of the wind is at right angles to the apparent wind, and that the cyclist must be able to ride at twice the speed of the wind.

The answer to the second part of the question is then 16 miles an hour.

The suggestion of a subtle distinction between the cyclist's best speed and his greatest speed seems unfortunate. That the cyclist never ventures out in a high wind, such as one blowing 50 miles an hour, and that he rides in a straight line, are matters of inference, presumably from the stops.

Compare Lord Burleigh's nod. C. S. J.

REVIEWS.

Stability in Aviation. By Prof. G. H. BRYAN, F.R.S. Pp. x+192. 5s. net. 1911. (Macmillan.)

The mathematical theory of aviation is rapidly being added to the lengthy list of subjects with which every mathematician should be acquainted. It is fortunate that the task of writing one of the first mathematical books on aviation has been undertaken by one who is not only a mathematician of first-rate eminence and a specialist in this particular line, but is also an experienced teacher. The reader will find the difficulties of the subject reduced to a minimum by the clearness of the exposition, and the care with which every assumption is emphasised, the notation explained, and the results summarised.

Prof. Bryan is justified in his claim that he has opened out to the student a valuable field for research, and he gives further assistance by adding a brief account of some work done by other authors, and by suggesting problems which await solution; though these would probably prove more troublesome than those attacked in this monograph. His suggestion that the subject might be suitable for junior University students is not so convincing; it would probably be beyond the powers of the average undergraduate. I must make an exception in favour of the "graphic statics of longitudinal equilibrium," which is quite elementary and will afford interesting and useful illustrations for the lecturer on Statics.

A flying aeroplane may be considered as a rigid body in steady motion under the action of air-pressure, gravity, and propeller-thrust, having a plane of symmetry which is vertical. If it is slightly disturbed, it may oscillate about its position of steady flight, in which case its motion is said to be "inherently stable." The purpose of the book is to investigate the conditions for this stability. It is easy enough to write down equations of motion which are universally applicable; and we find that the oscillations in the plane of symmetry can be conveniently separated from other oscillations; in other words, we may discuss separately longitudinal and lateral (or symmetric and asymmetric) stability. Difficulties arise, however, when we attempt to manipulate the equations of motion so as to obtain results of any interest; partly owing to the cumbrousness of the algebra involved, and partly because the forces acting on the aeroplane are unknown.

Prof. Bryan overcomes the difficulties by making certain assumptions; for instance, he supposes that the air-pressure on the planes is proportional to the square of the velocity, or he simplifies the algebra by taking the "angle of attack" as small. He is thus able to work out the conditions for longitudinal and lateral stability for all simple types of machine. The results can be extended to other types by making use of the "principle of independence of height" that longitudinal stability is only very slightly affected by raising or lowering the planes of a machine, and of the "principle of equivalence" that to every double lifting system of two narrow planes corresponds an equivalent single lifting system with the same conditions of equilibrium and longitudinal stability. For instance, the first principle shows that there is no practical difference between monoplane and biplane machines in respect of longitudinal stability.

Though the book will appeal mainly to the mathematician, the engineer will find the results suggestive; and he should have no difficulty in picking out what he wants without attempting to follow all the mathematical reasoning, especially as Prof. Bryan assists him with an excellent summary. For instance, the information that a machine loses longitudinal stability when rising, or gains lateral stability by the addition of two raised vertical fins, should be of great value and interest to him. The practical man, however, will in all probability raise one serious criticism. In order to arrive at his results the author is obliged to make a series of assumptions—that the air-resistances on the planes are linear functions of the small changes in linear and angular velocities; that in steady motion they are proportional to the square of the velocity; that they are normal to the planes; that they are proportional to sin a when the angle of attack a is small; that the pressure on an element of a narrow plane is independent of the motion of neighbouring elements, etc. Methods of approximation are also at times employed to simplify the algebra. The accumulated effect of small inaccuracies in each assumption may be considerable; and the results obtained in this way must be verified by experiment before the assumptions can be considered justified. The author does not work out the numerical result obtained by applying his formulae to any actual aeroplane; and even if he had pronounced any given machine stable or unstable, it might be difficult to verify the accuracy of his prediction; for inherent instability of the machine may be counteracted in practice by skill on the part of the aviator.

Probably the verification or modification of the assumption will be (as the author suggests) the task of the experimental physicist. At present there is a fear lest the mathematician may be brought to a standstill by the lack of data on which to work. This does not in the least diminish the debt which scientists owe to Prof. Bryan and his colleague, Mr. Harper, for providing them with the outlines of a theory which will doubtless be the basis of much future work, whatever improvements in detail may eventually prove necessary. H. Hilton.

The Hindu-Arabic Numerals. By D. E. Smith and L. C. Karpinski. (Ginn & Co.)

In this little treatise of 160 pages the results of recent and extensive research are given with an agreeable absence of dogmatic statement. In cases of doubt, and there are many such, the authors, while presenting the various theories in vogue, indicate to which in their own opinion the evidence inclines, but do not hesitate to confess that it is much less perfect than they could wish. They bring before us the struggle for existence of the set of symbols we now use, and enable us to realise how long that struggle was before the final victory was achieved, and how it was due not to any inherent superiority in the forms of the symbols, but to the introduction of a zero symbol and the invention of place value made possible by its existence. The subject should, of course, be of special interest to the mathematician, not only as dealing with the history of the tools he uses, but as bearing on the development of the very ideas now being examined with close philosophical scrutiny. But it is scarcely less interesting, and may possibly be more so, to the student of the general progress of civilisation, with which mathematical development is connected closely and in many peculiar ways.

The first four chapters discuss the history of our symbols and of other competing sets, the invention of place-value and the symbol zero. The remaining four discuss the introduction of our symbols into Europe and their gradual adoption, and includes a special chapter on the famous "Apices of Boethius" and the questions raised in connection with them.

About 600 references are given to previous writers, so that those who wish to pursue the matter further will have ample choice of study. These references are often accompanied by quotations of considerable length when the original is not easy of access.*

* Among the many ancient writings mentioned or quoted one may have special interest to some of the older members of the Association—"The Crafte of Nombrynge"—for proof copies of its reprint were, through the kindness of the Early English Text Society, distributed among members of the A.I.G.T. on the occasion of its preparation for publication, with a translation of the Arithmetic of John of Halifax and an early treatise on the use of the Counting Board.

Many reproductions of antique forms are given, including a photographic one of an inscription by the Buddhist King Asoka, and a table from Mr. Hill's paper in *Archæologia*, 1910. We give two quotations.

(i) On the discussion of the Boethius question. "It is true we have no records of the interchange of learning, in any large way, between Eastern Asia and Central Europe in the century preceding the time of Boethius. But it is one of the mistakes of scholars to believe that they are the sole transmitters of knowledge. As a matter of fact there is abundant reason for believing that the Hindu numerals would naturally have been known to the Arabs, and even along every trade route to the remote west long before the zero entered to make the place value possible, and that the characters, the methods of calculating, the improvements that took place from time to time, the zero when it appeared, and the customs in solving business problems, would all have been made known from generation to generation along the same trade routes from the Orient to the Occident. It must always be kept in mind that it was to the tradesman and the wandering scholar that the spread of such learning was due rather than to the schoolman."

(ii) On the spread of the numerals in Europe. "From his (Mr. Hill's) investigations it appears that the earliest occurrence of a date in these numerals on a coin is found in the reign of Roger of Sicily, in 1138 A.D. Until recently it was thought that the earliest such date was 1217 A.D. for an Arabic piece, and 1388 for a Turkish coin. Most of the seals and medals containing dates that were at one time thought to be very early have been shown by Mr. Hill to be of comparatively late workmanship. There are, however, in European manuscripts numerous instances of the use of these numerals before the twelfth century. Besides the example in the Codex Vigilanus, another of the tenth century has been found in the St. Gall MS. now in the University Library at Zürich, the forms differing materially from those in the Spanish Codex. . . . It is of interest to add that among the earliest dates of European coins or medals in these numerals are the following: Austria, 1484; Germany, 1489 (Cologne); Switzerland, 1424 (St. Gall); Netherlands, 1474; France, 1485; Italy, 1390." E. M. LANGLEY.

Die complexen Veränderlichen und ihre Funktionen. By G. KOWALEWSKI. Pp. ii + 455. 12 m. 1911. (Teubner.)

This is a charming book, written not only with extreme clearness and precision, but also with a freshness and originality seldom to be found in books which purport to be elementary text-books. When I began reading it I was not acquainted with any of Prof. Kowalewski's writings, but I had not spent an hour over it before I went out and ordered his *Grundzüge der Differential- und Integralrechnung*, of which it is a continuation. I recommend it with confidence to all who are interested in the foundations of the theory of functions of a complex variable.

The book opens with a long chapter dealing with the algebra and geometry of complex numbers, and containing a good deal of elementary group-theory. There are sections, for example, on finite groups of linear transformations, on the equivalence of positive quadratic forms, and on the modular group.

The second chapter is short, but particularly interesting. Some classical results in *Mengenlehre* are proved, first for a sphere, and then for a plane whose points have a (1, 1) correspondence with the points of the sphere. We are then introduced to the notions of a *convex* set of points, and of the least convex set which includes a given set; and these conceptions are used for the purpose of extending the first and second mean-value theorems to integrals involving complex functions of a real variable. Finally Taylor's theorem is proved for such functions. All this work is presented in a way very novel in a text-book, and shows Prof. Kowalewski quite at his best.

The third chapter deals with the definition of functions of a complex variable, elementary properties of complex power-series, the logarithmic and exponential functions, and so on, and is on more familiar lines. In the fourth the notions of a *path*, a *rectifiable path*, and an *integral along a path* are discussed with extreme care, and applied to functions defined by power-series in z or $1/z$. The fifth is the most valuable chapter in the book, containing as it does about the best and the most complete discussion of Cauchy's Theorem that I have seen.

The theorem is first established for a rectangle, by a modification of Goursat's method due in substance to Prof. E. H. Moore. We then pass to a series of theorems concerning real functions of two real variables. A function $f(x, y)$ is said to have an *eigentliches Differential* if $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ exist, and

$$\frac{f(x+h, y+k)-f(x, y)-h\frac{\partial f}{\partial x}-k\frac{\partial f}{\partial y}}{|h|+|k|}\to 0$$

when $|h|+|k|\to 0$. This definition, it should be observed, is practically the same as that given by Dr. W. H. Young in his papers in the *Proceedings of the London Mathematical Society.* It is then shown that if u and v have differentials, and $\frac{\partial u}{\partial y}=\frac{\partial v}{\partial x}$, then $u\,dx+v\,dy$ is a differential, and Cauchy's Theorem is deduced by what in the long run amounts to an accurate statement of the classical "double-integral" proof. After this the theorem is extended from a rectangle to any "normal region": finally a more direct proof is given which applies directly to such a region. The contents of the rest of the chapter are of a more ordinary type. But we realise the thoroughly modern character of the book when we find integral equations introduced in connection with the "Randwertaufgabe" for a circle.

I have left myself no space to say much of the last two chapters (theory of series and products, simply and doubly periodic functions, Weierstrass's and Mittag-Leffler's theorem). But this part of the book, though consistently sound and good, does not exhibit the peculiar merits of the author in as striking a way as does the earlier part. It only remains that I should wish the book every success in England as well as in Germany. G. H. Hardy.

Estudio elemental de la prolongación analitica. Por Patricio Peñalver y Bachiller. (Tesis, Madrid, 1911.)

The author gives a careful account of certain well known regions of function theory. He makes no pretence to originality; but this thesis is interesting as showing an interest in modern mathematics in Spain. G. H. Hardy.

Leçons sur les principes de l'Analyse. Par R. d'Adhemar. Tome I. (Paris: Gauthier-Villars, 1911.)

This volume seems to me hardly worthy of the firm by whom it is published. It is slap-dash and inaccurate, and although it has some novel features, its originality is not of so striking an order as to justify an addition to the long list of *Cours* and *Traités* already on the market. Moreover, it is written in what is to me at any rate a very irritating style. French is admittedly the language best adapted for scientific purposes, and the French of the best French mathematicians is unrivalled for lucidity and charm. But it is a language which has the defects of its qualities: its lucidity can wear thin, and its nervous terseness become jerky. M. d'Adhemar's mathematical style is not like that of Picard or Goursat.

I should add a few criticisms of details to justify these remarks. M. d'Adhemar remarks in his preface: "peut-être l'ordre suivi n'est-il pas assez logique. Dans la théorie des intégrales doubles et des potentiels, je me sers de quelques propositions qui sont démontrées plus loin, dans le Chapitre X. Mais l'exposition paraît ainsi moins lourde..." And at the beginning of Chapter VII. (Les Potentiels) we are told that "dans ce Chapitre, nous admettons les règles de dérivation des séries et des intégrales, théorie qui sera faite, en détail, dans le Chapitre X." What one wants in particular is certain theorems concerning the differentiation of multiple integrals. Now these theorems are not contained in Chapter X. (or elsewhere in the book). Only simple integrals are considered there. Moreover, the discussion in Chapter X. is erroneous. M. d'Adhemar gives as sufficient conditions for the truth of the equation,

$$\phi'(a)=\frac{d}{da}\int_a^b f(x, a)\,dx^2\int_a^b \frac{\partial f}{\partial a}\,dx,$$

the conditions (i) that $f'a$ is an integrable function of x, and (ii) that f is a continuous function of x and of a. "On n'a qu'à former $\phi(a+h)-\phi(a)$ et à passer à la limite." If he had taken the trouble to do this he would, of course, have found his conditions inadequate. And when he wishes to extend the result to the case of an infinite upper limit, he gives a faulty definition of the uniform convergence of an integral. A little later on he suddenly introduces a new symbol $\sim$ (without any explanation that I can find), and argues with it in the most casual way. In fact, all this part of the book is inaccurate and ill-digested. And even when it is impossible to say definitely that there is a mistake, the argument is often presented in such a form that the reader's confidence in the author is destroyed.

The best parts of the book seem to me to be the chapter on determinants and linear equations, and some of the sections dealing with integral equations. The parts about double and repeated integrals ought to be good, for the author has had the advantage of seeing the proofs of M. de la Vallée Poussin's new edition of his admirable *Cours*. It is a pity that the account which he gives of M. de la Vallée Poussin's work is not more adequate. G. H. HARDY.

Elemente der Funktiontheorie. Von Dr. NIELS NIELSEN. Pp. 520. Gr. 8vo. Price 15 marks. Weight $2\frac{1}{2}$ lb. 1911. (Teubner.)

This volume reproduces a set of lectures given at the University of Kopenhagen. By keeping strictly to the elements, Professor Nielsen is able to cover a wide field, so that Parts I. and II., on functions of real and complex variables, introduce the reader to a variety of aspects of these subjects. Part III. gives a detailed application to the elementary functions (including the gamma function) of the theory set out in the first two parts. The book is written with very great care. Accuracy is never sacrificed to brevity, and space is devoted to details of argument rather than to general discussion. This results in a certain monotony of style, especially in Part I., where it is not always easy to grasp the special point of each of the innumerable applications of the δ and ϵ method. A feature which adds greatly to the value of the book is the inclusion in nearly every paragraph of numerous exercises and references. H. P. H.

Théorie des Fonctions Métaspheriques. By N. NIELSEN. Pp. 208. 4to. 1911. (Gauthier-Villars.)

In Part I. there is collected the elementary theory required later, from the definition of a limit to transformations of gamma functions. Part II. begins with the definition of a general metaspherical function by means of two recurrence formulae, from which is immediately deduced Legendre's differential equation of second order. This function is shewn to include as particular cases all the hypergeometric functions of which use is made in analysis. The remainder of this part gives detailed information about the general function and others connected with it; among the properties discussed are further recurrence formulae, asymptotic expansions, and developments valid in various regions of the plane. Part III. is devoted to infinite series; in particular, any analytic function is expanded in a series of products of metaspherical functions, convergent on and within an ellipse. Part IV. deals with expressions in the form of definite integrals.

Owing to the very general nature of the functions treated, the algebra is heavy, and the masses of formulae are not easy to grasp. There is no discussion of results that would appeal to a general mathematical reader, but these advanced researches will be of great interest to those who have already made themselves familiar with the whole theory of Legendre and allied functions as it existed previously.

H. P. H.

Mémoire sur l'attraction du parallélépipède ellipsoidal. By M. DE SALVERT. 7 fr. 1908. (Gauthier-Villars.)

The object of the memoir is to calculate the total attraction of a homogeneous solid bounded by six confocal quadrics which can be arranged in pairs belonging respectively to the three different types. In order to simplify the work, the attraction is calculated at a point on the axis.

The author uses a system of curvilinear coordinates in which the cartesian coordinates (x, y, z) of a point on the quadric

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}=1$$

are expressed in the form

$$x=\sqrt{a^2-b^2}\quad Sn(u, k)dn(v, k')cn(w, k''),$$
$$y=\sqrt{b^2-c^2}\quad Sn(v, k')dn(w, k'')cn(u, k),$$
$$z=\sqrt{c^2-a^2}\quad Sn(w, k'')dn(u, k)cn(v, k'),$$

where $$k^2=\frac{a^2-b^2}{a^2-c^2},\quad k'^2=\frac{b^2-c^2}{b^2-a^2},\quad k''^2=\frac{c^2-a^2}{c^2-b^2}.$$

The work is highly analytical, Jacobian elliptic functions being used throughout. H. Bateman.

Geometria Descrittiva. By G. Loria.

The name "Descriptive Geometry" is often used rather loosely, and sometimes confused with Projective Geometry, but in the above treatise it is used exclusively to denote the representation of points in space by points in a plane. In fact, the book applies the methods which the draughtsman uses for his practical problems to the solution of theoretical questions. The work is unlike most English text-books, inasmuch as the writer does not aim at giving a detailed, manipulative knowledge of the subject, but aims rather at introducing general ideas by applying them to very simple cases. The method of the French mathematician, Monge, receives most attention. Central Projection is explained. In Part I. figures bounded by lines and planes are dealt with; in Part II. curves; in Part III. surfaces. Only the simplest cases are treated throughout, *e.g.* polyhedra, surfaces of rotation, the helix and the helicoid, cones and cylindrical surfaces, etc. In this way the student can acquire very clear notions as to loci, envelopes, tangent lines and planes, developable surfaces, without having to learn in the process detailed accounts of the properties of difficult species of such curves and surfaces. There are no sets of examples, most of these being worked out in the text. The work can be thoroughly recommended as an easy and lucid introduction to the notions of higher geometry. It is small in size and not expensive, and the general get-up of the book is quite satisfactory though not equal to that of many English text-books. William P. Milne.

Taschenbuch für Mathematiker und Physiker. Pp. 567. 7 mks. 1911. (Teubner.)

This valuable work, which now appears for the second time, contains an immense amount of information; and by the skilful condensation of matters which are to be found in every text-book, room has been provided for a reasonable amount of detail as to many interesting topics which are less frequently handled. Among these a biography of Minkowski (with portrait), twenty pages by Knopf on the determination of cometary orbits, a sketch of the theory of aggregates, a note on integral equations, and a note (ten pages) by Auerbach on the theory of relativity may be mentioned as typical and noteworthy. We hope that the publication will obtain a success proportional to the labour expended on it. C. S. J.

Sommations par une formule d'Euler. By Enrique Legrand. Pp. 46. 1911. (Buenos Ayres, Coni Hermanos; and Paris, Gauthier-Villars.)

The formula is the Euler-Maclaurin sum formula, involving the Bernoullian numbers,

$$\int_b^a f(x)\,dx=h\sum_a^b f(x)-\frac{h}{2}[f(b)+f(a)]-\frac{h^2B_1}{2\,!}[f'(b)-f'(a)]-\frac{h^4B_2}{4\,!}[f'''(b)-f'''(a)],$$

and by means of it the author effects the summation, exactly or approximately, of numerous series, and investigates certain limits, noting incidentally a numerical erratum in Bertrand's *Calcul Intégral*. A collateral feature of interest is that the text is printed in French and Spanish in parallel columns—the formulae being printed once for all the full width of the page. This is an excellent aid towards picking up a few words of mathematical Spanish. C. S. J.

Junior Algebra, with Answers. By W. G. Borchardt. Pp. viii+317. 2s. 6d. 1911. (Rivington.)

This will be found a very useful book. The exercises are good and so are the test papers.

The author is not at his best when he attempts a general explanation. Art. 1 is a case in point; or again, "In problems we are given statements expressing relations between known and unknown quantities." The definitions are not always good; *e.g.* "An Algebraic expression is a combination of symbols involving signs of operations." "Involution" is a continued product, but "evolution" is an operation, of which the reader is apparently left to discover the meaning.

The general arrangement is satisfactory, except that negative quantities are (in our opinion) introduced too early.

The factorization of $6x^2-19x+10$ is very cumbrous, and unnecessarily so; for the numbers 4, 15, whose product is 6 . 10 and sum 19 can at once be guessed by any student who has worked through a set of exercises like $x^2-19x+60=0$. Hence

$$6x^2-19x+10=6x^2-4x-15x+10=(3x-2)(2x-5).$$

Besides, the latter method has the advantage of showing, in reverse, the steps taken in expanding the product $(3x-2)(2x-5)$.

The "long rules" for square root and H.C.F. are introduced without any attempt at justification. This, we think, is a great mistake. These rules are not needed until the pupil has progressed considerably farther in Algebra than the point at which he is able to understand the justification, and the rules should be deferred to that point.

The pupil is frequently expected to jump at conclusions too readily. Thus, after plotting seven points on each of the lines $y=2x$, $y=3x+2$, he is expected to admit that an equation of the first degree represents a straight line.

Again $2x^3-3x^2-5x+8$ is divided by $x-2$, and it is found that the remainder is 2: this is found to be the result obtained when 2 is substituted for x in

$$2x^3-3x^2-5x+8.$$

On the strength of this one instance—by the way the reference should be Ex. 2, Art. 109, instead of Ex. 2, Art. 108—the pupil, in Art. 110, is supposed to admit the truth of the Remainder Theorem. Graphs appear to be overdone. In Art. 101, the graph of $3x^2+4x-3=0$ is used to find when this function is a minimum. The result is *verified algebraically*. The *approximate* graphical method takes a page and the *exact* algebraical verification eight lines!

In interpreting x^0, it is correctly stated that the result is true for all values of x except zero. It would have been better if the explanation were given that the method fails when $x=0$, because of the division of each side of the working equation by x^n. This is important in view of the interpretation of x^{-2} say when $x=0$. Is $0^{-2}=1/0^2=\infty$? or what is it? For the result is obtained by showing that $x^n\times x^{-n}=x^0$, and x^0 has no meaning attached to it when $x=0$.

It seems to be taken for granted that if $A \propto B$ when C is constant, and $A \propto C$ when B is constant, then $A \propto BC$; Ex. 3, Art. 139, being very loosely stated.

In Fractional Equations there is a bad error. Ex. 1, Art. 141, gives $x=-1$ as one of the answers to the equation $\frac{x+2}{x-1}+\frac{2(1+4x)}{1-x^2}+\frac{3}{x+1}=8$. It is not pointed out that the solution, which produced $x=-1$, fails in this case, for the step "multiply by x^2-1" is inadmissible when $x=-1$, hence $x=-1$ is not a solution. In fact the condition that all steps must be reversible is nowhere mentioned.

The book is well got up, the type being extremely readable. J. M. Child.

A School Algebra. Part II. By H. S. Hall. Pp. 301-450. 1s. 6d. 1911. Parts II. and III. Pp. 302-550. 2s. 6d. 1912. (Macmillan.)

These volumes are written in the excellent style one expects from this author. The general arrangement is good, and the pupil should be able to work through the exercises with very little help. By doing this, he will acquire considerable manipulative skill, but he is not likely to get a real grasp of the *principles*. For difficulties are shirked, particularly in the treatment of surds, logarithms, ratio and proportion. The terminology is not altogether satisfactory: for instance,

"quantity" is used for "number," and we meet with friends of the past in "incommensurable," "arithmetico-geometric series," "triplicate ratio," and others that we thought long dead.

Often the definitions are not in accordance with common usage or else they are too loose. Thus, the "sequence" 1, 3, 9, ... is called a "series." The author does not show the distinction in meaning between "irrational," as referring to the *character* of a number, and "irrational," as referring to the *form* of an expression or function. From the definition in Art. 354, it follows that $x+\sqrt{2}\,.\,y$ is an irrational function of x and y. From Art. 360, it is doubtful, according to the way in which the term is defined, whether the "simplest form" of $\sqrt{32}$ is $2\sqrt{8}$ or $4\sqrt{2}$.

The cart is before the horse in defining mantissa and characteristic. The second example in Art. 406 is very unfortunate; not only does it suggest that four-figure logarithms give results true to four significant figures, but the logarithm of 0·02748 is given as $\bar{2}{\cdot}4391$ instead of $\bar{2}{\cdot}4390$ to four places, and yet the result is correct! What would have happened if instead of a cube root we required a fifth power?

"Less than" is loosely used for "numerically less than." Thus in Art. 331, "Suppose r is less than 1, so that r^n diminishes as n increases"; a similar error occurs in the answer to Ex. XXIX. g, 10. We are told that "any *quantity* with zero index is equivalent to 1." Is zero a *quantity*? If so, is $0^0=1$? In Art. 390 a double meaning is assigned to $\sqrt{9}$ and a single meaning to $9^{\frac{1}{2}}$: yet we are told $9^{\frac{1}{2}}=\sqrt{9}$.

The faulty notion of a tangent is to be perpetuated:—"When ϕ moves up to P, and *ultimately coincides* with it, the line $P\phi$ becomes the tangent at P." Again, the gradient of $y=x^3$ at (2, 8) is found by putting $h=0$ ("h is very small and *ultimately vanishes*"), although this renders the reckoning meaningless.

Too many of the examples are of the old-fashioned examination type. For instance, "Find the ratio compounded of the duplicate ratio of 3 : 7 and the ratio 35 : 27." Again, "If y varies directly as x, and $x=60$ when $y=28$, etc." Most of the examples worked out in this chapter are of the foregoing type, and many at the beginning of the exercise are such that the variables have no reference to any definite problem.

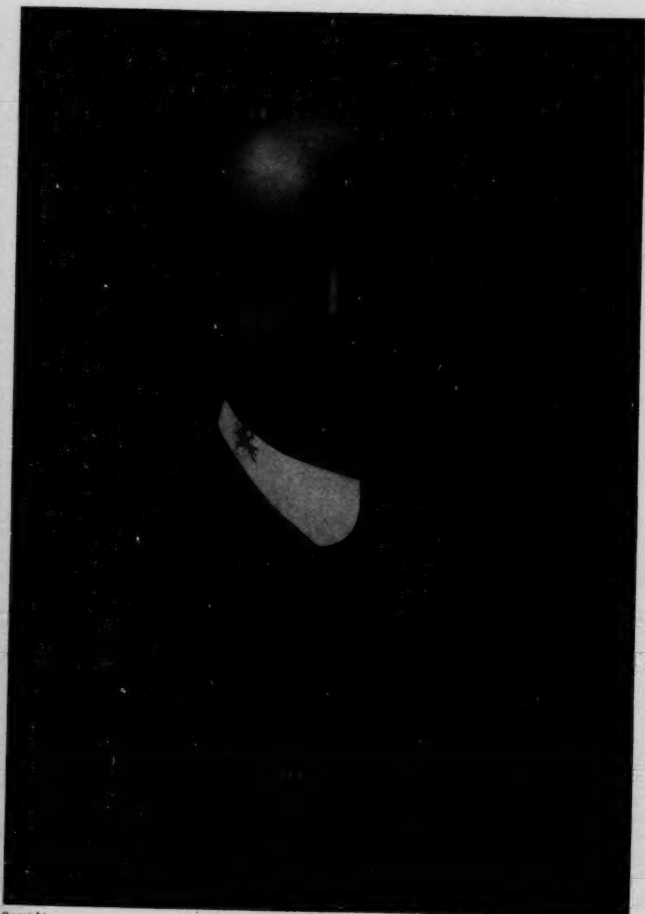
It may be argued that these are all small points, but such slips in treatment can only conduce to slackness in the students using the book; and I hold that the greater the reputation and following of an author, the heavier the responsibility that rests upon him to tell the truth, the *whole* truth, and nothing but the truth.

Part III. opens with Permutations and Combinations; then follows a short chapter on Mathematical Induction as a preliminary to the proof of the Binomial Theorem: we doubt if the proof of this theorem is so good as the shorter proof given in the author's *Higher Algebra*. The word "limit" is introduced on p. 483 without any other explanation than a reference to Art. 331, where the limiting value of an infinite G.P. is discussed; the examples on this section, however, are excellent. The methods in Partial Fractions are liable to lead to a good deal of arithmetical work which might be curtailed, and the point, noted at the bottom of p. 496, that the ordinary forms adopted for partial fractions are the only admissible forms, might have been insisted on further. An excellent account is given of the use of the exponential and logarithmic series, the proofs being deferred; a very useful chapter on the solution of miscellaneous types of equations by some special artifice concludes the text. Sufficient revision papers and miscellaneous exercises are given to render this book of great service as a school text.

J. M. CHILD.

The Theory of Determinants in the Historical Order of Development. By T. MUIR, F.R.S. Vol. II. The Period 1841 to 1860. Pp. xvi+475. 17s. net. 1911. (Macmillan.)

Dr. Muir has advanced a further stage on the road to completion of his great task of setting forth the calculus of determinants in the order of its development. He has now reached the beginning of a new epoch, that in which Sylvester continues his manifold activities, and another giant appears upon the scene in the person of Cayley. The first paper in the volume, indeed, is "the first fruits of Cayley's genius," a paper upon "A Theorem in the Geometry of Position," in



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which, for the first time, we see the new notation, the pair of upright lines. By their means, "every determinant became presentable, no matter how heterogeneous or complicated its elements might be; and the most disguised member of the family could be exhibited in its true lineaments." Dr. Muir happily illustrates the importance of the new notation by analogy:—"The infinitesimal calculus supplied with Lagrange's notation for the differential coefficient of $\phi(x)$, but unable to symbolise the differential coefficients of such a special function as $ax+bx^2$, or $\log(1-x)/(1+x)$, would be in the exact predicament of the theory of determinants prior to Cayley." Nearly fifty papers from the pen of him who was called by Mr. Glaisher "the greatest living master of Algebra" are referred to in this volume. Here we find analyses of the first papers on "gauche" or skew determinants—a product of the Cayley-Sylvester period—and various contributions dealing with orthogonants. Cayley's notes and papers teem with illustrations of the value of determinants as a weapon in the attack upon problems in which he happened to be interested from time to time. The "ever-to-be-lamented and commemorated" Sylvester, to use the epithets which he himself applied to Horner, is still, of course, to the fore, by the side of the friend whom he described as "the central luminary, the Darwin of the English school of mathematicians." What, with Sylvester's misprints, cases of "inaccuracy," "carelessness," and hasty remarks, etc., Dr. Muir has had more than one puzzle to clear up for the benefit of his readers. Salmon and Spottiswoode are responsible for about a dozen notes or papers apiece, but of other mathematicians in these isles, the average is only one or two. Brioschi, Faà di Bruno, and Bellavitis are to the front among the Italians, and Hermite is the only Frenchman, whose name occurs more than half-a-dozen times. Hesse and Jacobi, close rivals, bear away the palm among the Germans for quantity and quality. In the next volume we shall see the name of Clebsch appearing more than the once that we find it here. Oddly enough, Weierstrass does not seem to appear in either volume.

The divisions of the book are as follows:—Determinants in General (1841); Axisymmetric Determinants (1841); Alternants (1832); Compound Determinants (1850); Recurrents (1841); Wronskians (1838); Jacobians (1841); Skew Determinants (1846); Orthogonants (1841); Persymmetric Determinants (1841); Bigradients (1844); Hessians (1841); Circulants (1846); Continuants (1853); with various less common special forms. The date of the first paper in each section we have placed in brackets.

Dr. Muir's two volumes must not be taken to be mere bibliographies. The aim and scope of each contribution to the subject is clearly stated, its place in the general advance is duly noted, and such critical remarks as seem justified are added. He is singularly clear in his expositions, and writes with a judgment and impartiality which is beyond praise. Being so far from any of the great libraries of the world, he has had to work under difficulties which would have daunted less determined investigators. This substantially increases the debt that is owed to him by the mathematicians of his generation. We can only repeat what we said in the *Gazette* of the former volume, that it is a piece of work of which British mathematicians need not be ashamed.

It may be worth while adding that the difficulties experienced by Dr. Muir in obtaining access to sets of current or extinct mathematical periodicals have induced the Council of our Association to undertake the task of compiling a list of the various libraries throughout the country, with names and the dates, etc., of the various mathematical periodicals which they contain. Part of the author's precious leisure, while on a visit to this country, was wasted, owing to the fact that certain journals had to be traced from library to library—some at last being found in the hands of individuals. It is hoped that this action of the Council will prove of great advantage to mathematicians engaged in original research.

A Treatise on the Analytic Geometry of Three Dimensions. By G. Salmon. Fifth edition. Revised by R. A. P. Rogers. Vol. I. Pp. xxii+470. 9s. 1912. (Longmans, Green.)

The new edition of this classical work is due to the initiative of the Board of Trinity College, Dublin. It was inevitable that in the process of time it would be necessary to make additions to what used to be known to Dublin men and others

as "The Surfaces." Each edition grew larger, and the latest is no exception to the rule. The somewhat delicate task of deciding upon the new material for inclusion in the fifth edition has been wisely discharged by Mr. Rogers. Whatever was to be introduced, the additions must be such as would "fit in with the rest of the work." Sir Robert Ball (*Proc. London Math. Soc.*, Series II. Vol. I.) has pointed out a characteristic of Salmon's methods—they depended but little on the calculus. "The subjects that engrossed his attention, and which he made of such absorbing interest, were essentially treated by geometrical and algebraic processes." It is eminently characteristic of the man and his fertility of resource, that he would, as it were, carelessly toss to his readers additional proofs, *e.g.* "for the benefit of those who prefer a purely geometrical proof, one or two are added in the text." But, continues Sir Robert, "when he feels constrained to render some account of a subject like Monge's differential equation relating to curvature, he introduces it with a sort of apology as being a concession to some presumed desire on the part of the reader. . . . As one of his old pupils . . . once said to me, Salmon is not a calculus man." As the *Proceedings of the London Mathematical Society* is not within reach of all our readers, we venture to quote another short passage from Sir Robert's obituary notice: "A large part of the second half of this book . . . is devoted to most elaborate investigations . . . which, while they are marvellous expositions of mathematical ingenuity, transcend the powers of human conception almost as much as do the theorems of four dimensional space, and more than the most important investigations of non-Euclidean geometry. And here it may be noted that of four dimensional space or non-Euclidean geometry Salmon has never treated. It is, perhaps, too much to say that he regarded such speculations with scorn, but I have heard him say, jestingly, that he reserved such schemes for the next world." By type and brackets the new material is easily distinguishable from the old. We have four beautiful illustrations of models of quadrics: articles or paragraphs on the analytical classification of real quadrics, on projection and Fiedler's projective coordinates, on the non-Euclidean theory of distance and angle, and on the expression of twisted cubics and quartics by rational or elliptic parameters. "In differential geometry my aim has been to form a closer connecting link between Salmon's book and the more extensive and more purely analytical methods used by Bianchi, Darboux and others. I have, therefore, added articles on the now well-known Frenet-Serret formulae, with some applications on the intrinsic equations of a twisted curve, on Bertrand curves, and on the application of Gauss's parametric method to conformal representation, geodesic curvature and geodesic torsion. To the portion dealing with the differential geometry of curves on quadrics, I have added Staude's 'thread construction' for ellipsoids . . . and his definitions of confocal quadrics by means of 'broken distances.' . . . In the Golden Age of Euclidean geometry, analogues of these types were of great interest to men like Jacobi, MacCullagh, Chasles and M. Roberts, but Staude's constructions have virtually brought the subject to a conclusion." We have quoted sufficient from Mr. Rogers' preface to indicate the discretion he has exercised in the performance of his task. We have noticed but one misprint—"parumetric" (p. 423). On p. 454 the note reads oddly: "The intuitible meaning of Staude's theorems is most readily grasped by the use of models which have been constructed." Those who possess the New English dictionary may be interested in the history of the word "intuitible." Even in the early sixties many portions of "The Surfaces" called for "efforts of the geometrical imagination to which no adequate response can be made with present intelligence." Sir Robert Ball says, "We may be quite satisfied with the demonstration that on the reciprocal of the general surface of the nth order there will be a nodal curve of the degree $\frac{1}{2}n(n-1)(n-2)(n^3-n^2+n-12)$, but no finite intelligence can give geometrical vividness to such a statement." Nevertheless there will be no lack of veterans and numbers of the younger generation to look forward with pleasurable anticipation to the appearance of Vol. II.

2000 Théorèmes et Problèmes de Géométrie, avec Solutions. By A. Dalle. Pp. 825. 1912. (Ad. Wesmael-Charlier, rue de Fer, Namur.)

M. Dalle's Collection of Geometrical Exercises reminds us somewhat of the well-known *Exercices de Géométrie*, by F. J., continued by F. G.-M., published by Messrs. Mame of Tours or Poussielgue of Paris for the *Ecole des Frères Chrétiens*,

and now in a fifth edition. The resemblance, however, is carried little further than in the ground that is covered and the number of carefully selected examples. So far we do not seem to have felt the need for such a volume in this country. Altogether, the total number of examples worked out at length is just under one thousand. These the author divides into three categories: the "vénérables anciens" covered with the dust of the innumerable text-books in which they have been set and sometimes solved; examples notable for their connection with illustrious names, and for the frequency with which they are found in examination papers; and a considerable number of recent problems throwing a vivid light on certain properties or forming a link between other properties between which at first sight there seems to be no immediate relationship. Then there are recapitulary exercises containing original problems, some of considerable difficulty, which the compiler has considered of sufficient intrinsic interest to justify their inclusion. Of the "Recent Geometry of the Triangle" we find here and there a few remarkable properties, but no detailed exposition is given, inasmuch as the subject is outside the ordinary course for secondary schools. Homography and Involution are omitted, and Projection is but touched upon, on the ground that projective properties of figures do not become really stimulating until they are applied to lines and surfaces of the second order. In many cases alternative proofs are given, and suggestive notes and remarks are often appended to solutions. It is only occasionally that we find historical notes. The collection may be found useful for a scholarship boy to browse in, and the teacher will find it a useful quarry for "riders." It is well bound, on heavily sized paper, and is provided with good diagrams, those of solids being of more than usual excellence.

Bibliography of Non-Euclidean Geometry. By D. M. Y. Sommerville, M.A., D.Sc. Pp. 403. 10s. net. 1911. (Harrison & Sons, for the University of St. Andrews.)

Dr. Sommerville is to be congratulated on the completion of his laborious task of forming an accessible guide to the literature of the extended conception of space. The subject index contains some 600 headings and sub-headings, with over 5000 references. The total number of titles is more than 4000: theory of parallels, 700; non-euclidean geometry and the foundations of geometry, 1600; dimensions, 1800. The author index contains over 1200 names, among which we note that 460 are those of Germans or Austrians, 230 of French, Belgians and Swiss, 160 of British and 90 of Americans. The compiler has excluded works on the foundations of mathematics or arithmetic except in so far as they have explicit reference to geometry. Finite geometries and galois fields with non-archimedean geometries and algebras might have been included, but this ground has already been covered by Dr. Macfarlane. The reader is similarly referred to the treatise of Dr. and Mrs. Young for the bibliography of point groups. It has taken Dr. Sommerville nine years to complete his undertaking, and he has spared no effort to make it of use to those who are engaged in research. He has even added, wherever he could procure them, the prices of independently published works. The reading of the proofs seems to have been performed with meticulous care. We need say no more than that this bibliography is an invaluable contribution to the resources of the investigator. For a list of useful additions, etc., the reader would do well to consult Prof. Archibald's interesting and erudite notice of this work in *The Bulletin of the American Mathematical Society*, vol. xviii. No. 5, Feb. 1912.

Elementary Integrals: A Short Table. Compiled by T. J. I'A. Bromwich, F.R.S. 1s. net. 1911. (Macmillan.)

Many students will heave a sigh of gratitude to Mr. Bromwich for the relief afforded to them in the early stages of their physical or mathematical work. In less than 40 pages we have arranged for ready reference all the usual standard integrals and formulae of reduction. Examples of simple pseudo-elliptic cases, Simpson's formulae, and Planimetric formulae, with examples to be worked out, are included in addition to what is usually found in collections of this description. The sections on Irrational Algebraic Integrals are models of judicious condensation, and throughout the text one finds the touch of a master-hand.

Fergusson's Percentage Unit of Angular Measurement, with Logarithms ; also a Description of his Percentage Theodolite and Percentage Compass (for the use of Surveyors, Navigating Officers, Civil and Military Engineers, Universities and Colleges. By J. C. FERGUSSON). Pp. lxvii + 467. £3 3s.

Draw the four quadrantal lines of the circle inscribed in a square. Divide the circle into octants by the diagonals of the square. The sides of the square are tangents to the 8 octants. Divide each of the 8 tangent lines into 100 equal parts; subdivide the octant arcs into 100 parts by radii to the divisions of the tangents. Each division of the octant will subtend both 1/100 of the tangent "and a space equal to 1/100 of the radius." From the end of each quadrantal line number all the percentage divisions of the two adjacent octants in opposite directions from 0 to 100. Then every angle formed by a quadrantal line and any other inclined line drawn from the centre of the circle through one of the octant divisions will be a percentage angle, and the "tangent or perpendicular of that angle will be a measure of the departure between the two lines. Moreover, the amount of this departure will always be that percentage of the radius which is clearly marked by the number of the percentage angle itself." Here "perpendicular," "tangent," and "departure" are equivalent terms. The mind is now able to conceive of the amount of divergency between any inclined line and its base opposite any desired point on that base. It will be noted that we cannot tell offhand the departure contained in an angle of 40°, say, at a point on the base line 380 ft. from the angular point. But in the case of an angle of 40 % we can tell it at once, for 40 % of 380 ft. is 152 ft.

This is Mr. Fergusson's description of his percentage circle, the merit of which largely consists in the manner in which any angular surveying instrument to which it is attached is at once converted into a telemeter so that distances and levels may be found without chaining from the instrument station. His Percentage Theodolite is fitted with circles which can be read in this new system and, when required, in the old. No traverse tables are required, nor are tables needed for the reduction of the rod for inclined sights. All problems are solved there and then in the field. In one setting of the instrument the engineer, without tables or chaining, can determine the direction, distance, latitude and departure of the course, with the difference in levels between the stations. Nor are curve tables required for the ranging of curves of any radii. Further, improvements have recently been made in the instrument since 1905, when it was characterized by Prof. Heath as "extremely convenient in use," for "all the information about any point of the field can be known while the surveyor is on the spot."

The author claims that his discovery of the percentage unit of angular measurement reduces plane trigonometry to simple arithmetic by a few direct rules, "and now that the logarithmic tables for its use are complete it brings into life a new mathematical force." For 19 years he was working at the millions of calculations required for his purpose, and here, in 466 well set out and clearly printed pages, we have the logarithms. A few pages are devoted to a rough survey of the "Progress of the Mathematical Sciences and of the Men who advanced them." They seem somewhat out of place, and are too sketchy to be of much use. Moreover, they contain curiosities such as:—Duophantes of Alexandria, Albalegni, Regimontanus, Kemer, Rudolphene, Copernician, Lucascian Professor, etc. One or two of the dates given in these sections are open to question. It is not easy to see why the surveyor or the engineer need be told of Leibniz that "in 1710 he published his Essay on Theodocia, The Goodness of God, The Liberty of Man, and The Origin of Evil."

Lectures delivered at the Celebration of the Twentieth Anniversary of the Foundation of Clark University, under the auspices of the Department of Physics. By VITO VOLTERRA, ERNEST RUTHERFORD, ROBERT WILLIAMS WOOD, and CARL BARUS. Pp. 161. 10s. net. n.p. 1912. (G. E. Stechert & Co., 2 Star Yard, Carey Street, W.C.) The edition is limited to 500 copies.

The subject-matter of the lectures, delivered under the circumstances expounded in the title of this volume, being for the most part somewhat beyond the scope of the *Gazette*, we must confine ourselves to the mere enumeration of the subjects treated within its covers. Prof. Carl Barus, the distinguished physicist of Brown University, took as his subject, "Certain Physical Properties of the Iron Carbides,

together with Inferences deducible therefrom." Professor Wood, of the Johns Hopkins University, discoursed upon the "Optical Properties of Metallic Vapours," while Prof. Rutherford was at home in expounding the "History of the Alpha Rays from Radio-active Substances." Dr. Vito Volterra, Professor of Mathematical Physics at the University of Rome, took as his text, "Quelques progrès recents de la physique mathématique." In the first lecture we are shown the relations between the equations of electro-dynamics and the calculus of variations. The second is devoted to the consideration of the theory of elasticity, concluding with useful sections on the existence theorems, the method of successive approximations, Fredholm's theorem, etc. And the third, which to many will be of the greatest interest, deals with the general problem of elasticity and its relations to the phenomena of "heredity," *i.e.* to phenomena such as hysteresis and the like. The division of mechanics into the mechanics of heredity and the mechanics of non-heredity is due to Picard. The latter is that in which the future of a system depends at a given moment upon its present state and that which preceded it at an infinitely small period of time. The former is that in which each action leaves an "heritage" in the system, and the present state of the system depends on the whole of its previous history.

A College Text-book of Physics. By A. L. Kimball. Pp. ix + 692. 1912. (Bell & Sons.)

The justification offered by Prof. Kimball for adding to the number of text-books on the market consists very largely in his attempt to discuss the rudiments without the aid of those symbols and formulae which, as all the world knows, are repellent to the minds of the young. To the ordinary student he finds that algebra is "not a native tongue," and that if he is allowed to form a picture, "a sort of picture," of the conditions, he is able to grasp ideas more readily and to hold them more tenaciously than if he attempts to reach this goal by the aid of a chain of algebraical deduction. It must be remembered that the book is, in the first place, written for the American student taking the first year college course, and that an American "college" is not always what we call a "college" in this country. We cannot help feeling that, if it had been the lot of Prof. Kimball to spend the last few years in one of our secondary schools, he would have found that boys taught on modern lines are quite capable of following such chains of mathematical reasoning as are required in the exposition of elementary physics, and of grasping "the significance of the whole when they reach the end." It is not within the scope of the *Gazette* to enter upon the merits of a work on physics, and it will suffice to say that British teachers might well take the opportunity of looking at this excellent text-book. It is clearly written, adequately illustrated, and sufficiently up to date for students at the preliminary stage. The author has found it possible, on the whole, to avoid the dreaded mathematical formulae, but it is not surprising that he is unable to escape entirely from the fatal attraction, especially in the treatment of magnetism. Perhaps the highest praise we can give to Prof. Kimball is that he has fairly substantiated his claim that he has aimed at giving the reader some insight into the "underlying unity of the subject."

THE LIBRARY

The Librarian acknowledges with thanks the receipt of a number of Text-books presented to the Library by the Clarendon Press. Also the gift of a volume by Mr. C. A. Eves.

The Library has now a home in the rooms of the Teachers' Guild, 74 Gower Street, W.C. A catalogue will be issued to members in due course, containing the list of books, etc., belonging to the Association and the regulations under which they may be inspected or borrowed.

The Librarian will gladly receive and acknowledge in the *Gazette* any donation of ancient or modern works on mathematical subjects.

Wanted by purchase or exchange:

1 or 2 copies of *Gazette* No. 2 (very important).
2 or 3 copies of Annual Report No. 11 (very important).
1 or 2 „ „ Nos. 10, 12 (very important
1 copy „ Nos. 1, 2.

BOOKS, ETC., RECEIVED.

The Calculus for Beginners. By W. M. BAKER. Pp. viii + 166. 3s. 1912. (Bell & Sons.)

Public School Examination Papers in Mathematics. Compiled by P. A. OPENSHAW. Pp. 134. 1s. 6d. 1912. (Bell & Sons.)

Magnetism and Electricity, A Manual for Students in Advanced Classes. By E. E. BROOKS and A. W. POYSER. Pp. viii + 633. 7s. 6d. net. 1912. (Longmans, Green.)

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An Introduction to the Lie Theory of One-Parameter Groups, with applications to the Solution of Differential Equations. By A. COHEN. Pp. vii + 248. 5s. net. (Heath ; Harrap.)

Bollettino della Mathesis. April, 1912. Nos. 1-4. Pp. 58. With a Supplement. 4 l. per ann. (Cooperativa Tipografica Manuzio, Rome.)

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